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## 1. *The Field of an Electron at Rest and in Uniform Motion.* *By* H. BATEMAN (*California Institute of Technology*)*.

### § 1. *The Hypothetical Structure of an Electrostatic Field.*

WE shall assume that the usual ideas relating to the energy and momentum in an electromagnetic field hold good for the *simple radiant fields* in which the vectors E and H satisfy the relations

$$E^2=H^2, \quad (E.H)=0.$$

For the sake of generality, however, we shall multiply the usual expressions for energy and momentum by a constant factor $\lambda$ which, later on, will be put equal to $\frac{1}{3}$.

In these fields, then, energy and momentum flow along straight lines with the velocity of light, the direction of flow at any point being that of the momentum. Dynamically the state of affairs may be compared with the free flight of a system of particles which do not influence one another and are not under the influence of external forces. We may, if we wish to adopt a reasonable hypothesis, speak of these particles as *quanta* †, but to be on safer ground we shall call them *field-atoms*.

More complicated electromagnetic fields may be obtained

* Communicated by the Author.

† The idea of making quanta responsible for the energy and momentum of an electromagnetic field of this type and for the forces which it exerts may be regarded as due to Einstein. It was developed by Lorentz in his lectures at Pasadena.

by superposing simple radiant fields*; for instance, if we differentiate with respect to $z$ the components of E and H belonging to the simple radiant field

$$H_x = \frac{y}{x^2+y^2}, \qquad E_x = -\frac{xz}{r(x^2+y^2)},$$

$$H_y = -\frac{x}{x^2+y^2}, \qquad E_y = -\frac{yz}{r(x^2+y^2)},$$

$$H_z = 0, \qquad E_z = \frac{1}{r},$$

we obtain the components of the field vectors for a point charge of electricity at the origin. The electrostatic field of a point charge may clearly be obtained in an infinite number of ways by superposing simple radiant fields, but it does not follow that it should be regarded as made up permanently of field-atoms, for the simple radiant fields may affect each other in some way—the field-atoms may collide and form *field-molecules*.

An attempt to obtain some information regarding the distribution of these hypothetical field-molecules will be based upon some rather crude analogies.

In the theory of light-quanta the number of $s$-fold light-quanta (light-molecules) of frequency $\nu$ in a cubic centimetre of cavity radiation is given by the formula of Wolfke †,

$$n_\nu^{(s)} = \frac{8\pi\nu^2}{sc^3} e^{-\frac{sh\nu}{RT}}, \qquad \ldots\ldots (1)$$

where $n_\nu^s d\nu$ is the number whose frequencies lie between $\nu$ and $\nu+d\nu$. The associated density of energy is

$$u_\nu^{(s)} = \frac{8\pi h\nu^3}{c^3} e^{-\frac{sh\nu}{RT}}. \qquad \ldots\ldots (2)$$

Denoting the energy $sh\nu$ of a light-molecule by U, the number of $s$-fold light-molecules whose energies lie between U and $U+dU$ is

$$n_U^{(s)} = \frac{8\pi U^2}{s(shc)^3} e^{-\frac{U}{RT}}, \qquad \ldots\ldots (3)$$

while the density of energy is

$$u_U^{(s)} = \frac{8\pi U^3}{s(shc)^3} e^{-\frac{U}{RT}}. \qquad \ldots\ldots (4)$$

The field-molecules at present under consideration are not

* H. Bateman, Proc. London Math. Soc. (2) xviii. p. 95 (1919).
† M. Wolfke, *Phys. Zeitschr.* xxii. p. 375 (1921).

necessarily light-molecules *, and we cannot speak of a temperature of an electrostatic field ; but in the electrostatic field of an electron or proton there is a quantity, $\frac{e^2}{r}$, which has the same dimensions as RT, $e$ being the charge on an electron ; and so an attempt will be made to construct analogous formulæ in which RT is replaced by $\frac{\alpha e^2}{r}$, $\alpha$ being a constant which is at present undetermined.

Since, in the theory of the Bohr atom of hydrogen, the kinetic energy of the revolving electron is $\frac{e^2}{2r}$, it seems possible that

$$\tfrac{3}{2}\mathrm{RT}=\frac{e^2}{2r}.$$

$\alpha$ would then be $\frac{1}{3}$. This guess may be wide of the mark.

Pursuing our analogy, we shall assume that the number of field-molecules with energy lying between U and U$+dU$ in one cubic centimetre is

$$n_{\mathrm{U}}=k\mathrm{U}^2e^{-\frac{\mathrm{U}r}{\alpha e^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $k$ is some unknown constant factor having the same physical dimensions as $(hc)^{-3}$. The density of energy which lies between U and U$+dU$ is then †

$$\mathrm{W}_{\mathrm{U}}=k\mathrm{U}^3e^{-\frac{\mathrm{U}r}{\alpha e^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the total density of energy is

$$\mathrm{W}=k\int_0^\infty \mathrm{U}^3e^{-\frac{\mathrm{U}r}{\alpha e^2}}d\mathrm{U},$$

$$=6k\left(\frac{\alpha e^2}{r}\right)^4=6k\alpha^4e^6.\frac{e^2}{r^4}. \quad . \quad . \quad . \quad . \quad (7)$$

The total number of field-molecules is, on the other hand ‡,

$$\mathrm{N}=k\int_0^\infty \mathrm{U}^2e^{-\frac{\mathrm{U}r}{\alpha e^2}}d\mathrm{U},$$

$$=2k\left(\frac{\alpha e^2}{r}\right)^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

* A calculation seems to indicate that light-atoms and light-molecules alone would not give a sufficiently high value of the density of energy. A field-molecule is supposed to have stationary mass.

† The calculation mentioned in the last footnote is based on the supposition that $k=\frac{8\pi}{(hc)^3}\sum_1^\infty\frac{1}{s^4}$ and $\alpha=\frac{1}{3}$.

‡ This law is practically the same as one used in a previous attempt to determine the structure of an electrostatic field. 'Publications of the Astronomical Society of the Pacific,' April 1922

If we suppose that a certain percentage of these are moving outwardly in a radial direction with some velocity $v$ which is the same for all the molecules of this set, the rate at which these molecules diminish in number owing to collisions is proportional to

$$-\frac{1}{r^2}\frac{d}{dr}\left[2k\frac{\alpha^3 e^6}{r}\right] = \frac{2k}{r^4}\alpha^3 e^6 ;$$

it is thus proportional to the density of energy, and we have a law analogous to the law of mass action. On account of the inverse cube law, the number of field molecules of this type that go to infinity is zero.

If this hypothesis is correct, there is some mechanism in an electrostatic field which prevents the field entities from completely leaving the neighbourhood of the charge. Electromagnetic theory itself suggests that this may be the case, for the coordinated motion of the set of point charges constituting an electron of finite size appears in simple cases, such as that of uniform motion, to be of such a nature that lines of force issuing from different point charges are permanently incident *. Now it may be legitimate to regard the lines of force of a point charge as made up of field-atoms or molecules travelling with the velocity of light along straight lines † ; but the theorem just mentioned suggests that the structure of the electron is of such a nature that the emitted field-atoms (molecules) collide sooner or later, and it seems reasonable, then, to suppose that the electron uses the same entities over and over again in producing its field, so long as it is in uniform motion and free from the influence of external fields.

It may be possible to carry our analogy further, and to suppose that at any fixed distance $r$ from the centre of an electron there is a condition analogous to that of thermal equilibrium. The work of Pauli ‡ and Bothe § on thermal equilibrium between electrons and light-quanta can be extended perhaps to the present problem if it is assumed that the number of field-molecules with stationary mass lying between M and $\mathrm{M}+d\mathrm{M}$, and component momenta lying between $(\mathrm{G}_x,\mathrm{G}_y,\mathrm{G}_z)$ and $(\mathrm{G}_x+d\mathrm{G}_x,\ \mathrm{G}_y+d\mathrm{G}_y,\ \mathrm{G}_z+d\mathrm{G}_z)$ is

* H. Bateman, *l. c.*

† Leigh Page, Amer. J. Science, xxxviii. p. 169 (1914) ; H. Bateman, *l. c.* The field-atoms may move either towards or away from the charge.

‡ W. Pauli, *Zeitschr. f. Phys.* xviii. p. 272 (1923).

§ W. Bothe, *ibid.* xxiii. (1924).

proportional to

$$e^{-\frac{Ur}{ae^2}} dG_x dG_y dG_z f(M) dM, \quad . \quad . \quad . \quad . \quad (9)$$

the factor being a function of M. This is a simple generalization of Maxwell's law and of the equation adopted by Pauli * and Bothe †. It is not certain that this hypothesis is compatible with (6), and it is likely that a more general form will be needed if the stationary mass of a field-molecule can alter in a collision. The hypothesis makes the total momentum per unit volume equal to zero, and the supposition of a type of thermal equilibrium at the distance $r$ seems to require that there should be no transverse flow of momentum and, in fact, that the field should be entirely free from stress. Indeed an unbalanced pressure depending on $r$ would give rise to a body force in the field. The pressure of the field-molecules must be balanced by forces between them. A residual pressure or tension which is constant throughout space is quite compatible with the present ideas, but we are interested at present in the departure from uniform normal stress when we speak of stresses in the field. In this sense, then, we may conclude that the field is free from stress. This conclusion should hold whether equation (9) needs alteration or not, and must now be discussed.

In the classical electromagnetic theory there are stresses even in the field of an isolated electric charge, and their existence is supposed to be confirmed by the existence of a stress at the surface of a charged conductor or soap-bubble; but in the case of an electron the supposed existence of such a stress presents a difficulty, as the electron might be expected to blow up. If, on the other hand, we adopt the above idea that the electrostatic field of an electron is entirely free from stress, some modification must be made in the specification of the state of stress in an electromagnetic field. A possible modification has already been suggested but not definitely adopted ‡, and will now be presented in a more complete form.

## § 2. *Energy, Momentum, and Stresses in the Field of an Electron in Uniform Motion.*

Using W to denote the density of energy, $(S_x, S_y, S_z)$ to denote the components of the flow of energy, $(G_x, G_y, G_z)$ for

* W. Pauli, *l. c.* † W. Bothe, *l. c.*

‡ Mess. of Math. lii. p. 116 (1922); 'Science,' lvii. p. 238 (1923). The calculations in the first paper are made with a slightly different tensor, but the present one is mentioned as suitable.

the components of the density of momentum, and $X_x, X_y, X_z$; $Y_x, Y_y, Y_z$; $Z_x, Z_y, Z_z$ for the components of stress, we assume that

$$3X_x = E_x^2 + H_x^2 - \tfrac{1}{2}(E^2 + H^2) - 2\Psi\frac{\partial^2\Psi}{\partial x^2} + \Gamma + \frac{\partial^2\Omega}{\partial x^2} - \Box\Omega,$$

$$3X_y = E_xE_y + H_xH_y - 2\Psi\frac{\partial^2\Psi}{\partial x\partial y} + \frac{\partial^2\Omega}{\partial x\partial y} = 3Y_x,$$

$$3W = \tfrac{1}{2}(E^2 + H^2) + \frac{2}{c^2}\Psi\frac{\partial^2\Psi}{\partial t^2} + \Gamma - \frac{1}{c^2}\frac{\partial^2\Omega}{\partial t^2} - \Box\Omega,$$

$$3S_x = c(E_yH_z - E_zH_y) - 2\Psi\frac{\partial^2\Psi}{\partial x\partial t} + \frac{\partial^2\Omega}{\partial x\partial t} = 3c^2G_x,$$

. . . . . .

where

$$\Gamma = \left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2 + \left(\frac{\partial\Psi}{\partial z}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\Psi}{\partial t}\right)^2,$$

$$\Omega = \tfrac{5}{8}\Psi^2,$$

and

$$\Box\Omega \equiv \frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{\partial^2\Omega}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\Omega}{\partial t^2}.$$

In these equations $(E_x, E_y, E_z)$, $(H_x, H_y, H_z)$ denote the components of electromagnetic field vectors that are calculated from the volume density $\rho$ and velocity $v$ of the electricity by means of the usual formulæ

$$H = \operatorname{curl} A, \qquad E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\Phi,$$

$$\Phi = \frac{1}{4\pi}\iiint\frac{d\xi d\eta d\zeta}{r}[\rho]_{\xi,\eta,\zeta},$$

$$A = \frac{1}{4\pi c}\iiint\frac{d\xi d\eta d\zeta}{r}[\rho v]_{\xi,\eta,\zeta},$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = c^2(t-\tau)^2, \quad (\tau \leq t),$$

and $\Psi$ is a scalar potential which is invariant under the transformations of the theory of relativity (for uniform motions), and is given by the formula

$$\Psi = \frac{1}{4\pi}\iiint\frac{d\xi d\eta d\zeta}{r}\left[\rho\sqrt{1-\frac{v^2}{c^2}}\right]_{\xi,\eta,\zeta}.$$

Outside the electron we have $\Box\Psi=0$, and so the equations

$$\left.\begin{aligned}\operatorname{div} X-\frac{\partial G_x}{\partial t}&=0,\\ \operatorname{div} S+\frac{\partial W}{\partial t}&=0,\end{aligned}\right\}\quad\ldots\ldots\quad(10)$$

which express the principles of the conservation of momentum and energy, are seen to be satisfied.

When the electron is stationary we may write

$$\Psi=\Phi=\frac{\sigma}{r},$$

where $4\pi\sigma=e$, the charge on the electron. We have then

$$\Gamma=\frac{\sigma^2}{r^4},\qquad \Box\Omega=\frac{5\sigma^2}{4r^4},\qquad \frac{\partial^2\Omega}{\partial x^2}=\frac{5\sigma^2x^2}{r^6}-\frac{5\sigma^2}{4r^4},$$

$$E_x^2-\tfrac{1}{2}E^2=\frac{\sigma^2x^2}{r^6}-\frac{\sigma^2}{2r^4},\qquad 2\Psi\frac{\partial^2\Psi}{\partial x^2}=-\frac{2\sigma^2}{r^4}+\frac{6\sigma^2x^2}{r^6},$$

$$X_x=X_y=S_z=0,\qquad W=\frac{\sigma^2}{12r^4}.$$

*Thus there is no momentum and no flow of energy. The space outside the electron is entirely free from stress.*

Equating the present value of W to that obtained in § 1, we have

$$6k\alpha^4e^6=\frac{1}{12(4\pi)^2}.$$

The presence of the factor 3 in the expressions for $X_x$ etc. may puzzle the reader at first, but the explanation is * that in the present theory one-third of the electric force arises from the field vectors E and H, while the other two-thirds arises from the scalar potential $\Psi$.

The expressions for $X_x$ etc. have been constructed so that they will transform in the standard way in a relativity transformation †.

Passing to the case of uniform motion with the aid of the

* *L. c.* This cannot be regarded as definitely proved.

† Dr. Silberstein has mentioned to me in a letter that this point ought to be emphasized. The terms depending on $\Omega$ are analogous to those considered by E. Furlanetto in a solution of the problem of Morera (*Nuovo Cimento*, xvi. p. 139, 1918).

relativity transformation,

$$x=\beta(x'-vt'), \quad t=\beta\left(t'-\frac{vx'}{c^2}\right), \quad y=y', \quad z=z',$$

$$\beta=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}},$$

we find with the aid of the invariant relation

$$\mathrm{X}_x dx^2+\mathrm{X}_y dy^2+\mathrm{X}_z dz^2+2\mathrm{Y}_z dy\,dz+2\mathrm{Z}_x dz\,dx+2\mathrm{X}_y dx\,dy$$
$$+2\mathrm{S}_x dx\,dt+2\mathrm{S}_y dy\,dt+2\mathrm{S}_z dz\,dt-c^2\mathrm{W}\,dt^2=\mathrm{X}_x' dx'^2+\ldots$$
$$\ldots \quad -c^2\mathrm{W}'dt'^2,$$

that

$$\mathrm{W}'=\beta^2\mathrm{W}, \qquad \mathrm{S}_x'=c^2\mathrm{G}_x'=v\beta^2\mathrm{W},$$

$$\mathrm{X}_x'=-\frac{v^2}{c^2}\beta^2\mathrm{W}, \qquad \mathrm{X}_y'=\mathrm{X}_z'=0,$$

$$\mathrm{Y}_z'=\mathrm{Y}_y'=\mathrm{Z}_z'=\mathrm{S}_y'=\mathrm{S}_z'=0.$$

Hence the momentum is parallel to the axis of $x$, and is $\mu' v$, where the mass density $\mu'$ is given by

$$\mu'=\frac{\mathrm{W}'}{c^2}.$$

The energy and momentum, moreover, flow with velocity $v$ in the direction of the axis of $x$. Our tensor, $\mathrm{X}_x' \ldots \mathrm{W}'$, thus satisfies the conditions that many physicists have secretly thought* ought to be satisfied by a tensor which correctly represents the state of affairs outside the electron. It should be noticed also that $\mathrm{W}'$ is positive, $\mathrm{X}_x'$ negative, and $\mathrm{S}_x'^2<\mathrm{W}^2$; it is quite possible, then, for the densities of energy and momentum and the flow of momentum to be produced by moving field-atoms or field-molecules †.

By limiting our expressions for $\mathrm{X}_x \ldots \mathrm{W}$ to the case of the uniform motion of an electron, we avoid the difficulty that negative values of W appear in some places when an attempt is made to apply this tensor as it stands to the case of accelerated motion of an electron. That either the tensor or the field needs modification in this case appears also from the circumstance that the formula for the force between two electrons in relative motion is not quite right.

* Judging from a conversation with Drs. Lorentz and Epstein.

† In making this remark, I must acknowledge my indebtedness to some investigations of Einstein and Lorentz which emphasize the importance of these inequalities, which are well known in kinetic theory.

§ 3. *The Interior of the Electron.*

For the interior of the electron the expressions for $X_x$ etc. need modification, for the density of energy should be positive and the stress should vanish at the boundary when the electron is at rest. These conditions may be satisfied by adding to the former expressions for $X_x \ldots$ additional terms $X_x^a \ldots$, given by

$$3X_x^a = -2\omega\Psi_a - 2\Psi_a\beta\rho\frac{v_x^2}{c^2},$$

$$3X_y^a = -2\Psi_a\beta\rho\frac{v_x v_y}{c^2},$$

$$3W^a = -2\omega\Psi_a + 2\Psi_a\beta\rho,$$

$$3S_x^a = 2\Psi_a\beta\rho v_x,$$

where $\beta = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$, and where $\Psi_a$ and $\omega$ are the values of $\Psi$ and $\rho\sqrt{1-\dfrac{v^2}{c^2}}$ at the boundary of the electron, these values being supposed constant over the boundary. The conditions may be satisfied in another way by replacing the quantities $\beta$, $\rho$, $v$ in the foregoing expressions by their constant boundary values. This eliminates the last term in the ensuing expression for W and gives the usual expression for the total energy of an electrostatic field. The present calculations are made with the expressions given in the text, but it must be acknowledged that the alternate solution has some advantages.

When the electron is stationary we may write $\Psi = f(r)$. Using primes to denote differentiations with respect to $r$, we have now

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{x^2}{r^2}f'' + \frac{1}{r}f' - \frac{x^2}{r^3}f',$$

$$\frac{\partial^2\Omega}{\partial x^2} - \square\Omega = -\frac{4}{r}ff' - \frac{y^2+z^2}{r^2}\left[2f'^2 + 2ff'' - \frac{2ff'}{r}\right].$$

At the point $x=a$, $y=0$, $z=0$ on the spherical boundary $r=a$, $\Psi$ and $\Psi'$ are continuous, consequently $f.f'$, $\dfrac{\partial^2\Psi}{\partial x\,\partial y}$, $\dfrac{\partial^2\Psi}{\partial x\,\partial z}$, and $\dfrac{\partial^2\Omega}{\partial x^2} - \square\Omega$ are continuous, but the discontinuity in $\dfrac{\partial^2\Psi}{\partial x^2}$ is $-\omega$ and is just sufficient to make $-2\omega\Psi_a - 2\Psi\dfrac{\partial^2\Psi}{\partial x^2}$

continuous. Hence the stresses $X_x$, $X_y$, $X_z$ are all continuous at the boundary and are consequently zero. Turning the axes round, we see that *the boundary of the electron is entirely free from stress.*

Inside the electron

$$W = \tfrac{1}{12}f'^2 + \tfrac{5}{12}\rho f + \tfrac{2}{3}\Psi_a(\rho - \rho_a).$$

Now the law of variation of $\rho$ with $r$ is determined by equations (10), which give

$$\rho\Psi' = 2\Psi\rho',$$

or $\Psi = b\rho^2$, where $\rho$ is a constant. Since $\Psi$ is the electrostatic potential derived from $\rho$, we write

$$\rho = A_0 + A_2 r^2 + A_4 r^4 + \ldots,$$

$$\Psi = \frac{A_0}{6}(3a^2 - r^2) + \frac{A_2}{20}(5a^4 - r^4) + \frac{A_4}{42}(7a^6 - r^6) + \ldots,$$

and find that

$$\rho = A_0\left[1 - \frac{s}{12}\left(\frac{r}{a}\right)^2 - \frac{s^2}{720}\left(\frac{r}{a}\right)^4 - \ldots\right]$$

where $s = \dfrac{a^2}{A_0 b}$ is given by the equation

$$0 = 1 - \tfrac{1}{2}s + \tfrac{1}{48}s^2 + \ldots -.$$

This equation for $s$ appears to have two positive roots, the smaller of which is approximately

$$s = 2{\cdot}209012572.$$

It may be proved that the series for $\rho$ converges when $s$ has this value.

With either root of this equation, $|\rho|$ decreases from a maximum value at the centre of the electron to a smaller value $|\rho_a|$ at the boundary; also $\Psi$ and $\rho$ evidently have the same sign, and so $\rho f$ and $\Psi_a(\rho - \rho_a)$ are both positive.

*Hence* W *is everywhere positive.*

The mass $m$ of the electron, when calculated from its total energy, is given by

$$mc^2 = 4\pi\int_0^a \left[\frac{1}{12}f'^2 - \frac{5}{12}f\left(f'' + \frac{2}{r}f'\right) + \frac{2}{3}\Psi_a(\rho - \rho_a)\right]r^2\,dr + \frac{4\pi\sigma^2}{12}\int_a^\infty \frac{dr}{r^2}$$

$$= 4\pi\left[\frac{1}{4}r^2 ff' + \frac{1}{6}r^3 f'^2 + \frac{2}{9}r^3 f^{3/2}b^{-1/2} - \frac{2}{3}r^2 f'\Psi_a - \frac{2}{9}r^3\rho_a\Psi_a\right]_0^a + \frac{\pi\sigma^2}{3a}$$

$$= \frac{8\pi\sigma^2}{3a} = \frac{e^2}{6\pi a}.$$

*Hence the mass of the electron is the same as that of a Lorentzian electron with uniform surface charge.* The same result is obtained when the mass is calculated from the momentum of the electron in uniform motion *. A calculation shows that the new electron is stable for radial displacements in which the electricity acts as an incompressible and inextensible fluid.

The fact that the present analysis does not indicate the size of the electron need not trouble us, because the radius of an electron must be expressed in terms of something else and at present there is no other length available. When the problem of the motion round a nucleus of an electron of finite size has been solved, it will be possible to compare the radius of an electron with its distance from the nucleus. In Einstein's theory of gravitation the size of the electron is supposed to be determined by the curvature of the world.

§ 4. *Further Remarks on Collisions between Field-atoms.*

In § 1 it is supposed that two field-atoms may combine to form a field-molecule with a stationary mass which may be different from zero. Now, if the field-atoms are light-quanta, the question may be asked why field-molecules do not exist to any appreciable extent in cavity radiation. The explanation may be that in cavity radiation a light-quantum is orientated at right angles to its direction of motion †, while in an electrostatic field this may not be the case. The nature of a collision between two light-quanta may very well depend upon their orientations ‡, especially if light-quanta are

* This depends on the fact that the mean value of $X_x$ over the stationary electron is zero.

† According to Emden (*Phys. Zeitschr.* xxii. p. 513, 1921), a light-quantum has 6 degrees of freedom, and so we can speak of its orientation.

‡ Dr. Tolman has also had in mind the possibility of two light-quanta describing orbits round each other under certain circumstances and sticking together, so to speak. The idea of light condensing into some form of matter is used by Sir Oliver Lodge (Phil. Mag. xli. p. 549, 1921) and by W. D. Macmillan in several articles; it is probably a very old idea. The idea seems to imply that a light-quantum has an appreciable influence on the space-time metric. In this connexion it may be mentioned that an interval $ds$ satisfying Einstein's gravitational equations and representing a mass travelling with uniform velocity $c$ has been found to be given by

$$ds^2=c^2dt^2-dx^2-dy^2-dz^2+f(x-ct, y, z)(dx-cdt)^2,$$

where $\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0$ outside the moving mass. The idea that light produces a deformation of space but not of time is used by F. Famà, 'Le radiazioni energetiche,' Bologna, 1921; it must naturally lead to a different result. The formula just mentioned has been confirmed by Mr. H. P. Robertson, it was suggested by a result obtained by the author in 1910 (Proc. Lond. Math. Soc. (2) vol. x. p. 7).

analogous to electric dipoles. Furthermore, at a sufficient distance from the boundary the number of field-molecules per c.c. arising from the electrical fields of the material particles will be negligible in comparison with the number of quanta.

To make this difference in orientation seem plausible, let us consider the generation of a line of electric force by emitted field-entities that travel without colliding. If the motion of the point charge is uniform and rectilinear, the direction of projection of the entities does not vary, and so the line joining two entities, emitted at successive instants, keeps one direction and never becomes perpendicular to its direction of motion. When, however, the motion of the point charge of electricity is accelerated in any manner, the direction of projection of successive entities is not the same, and the line joining these entities varies during their motion and becomes eventually almost perpendicular to their directions of motion. This corresponds more nearly to the case of the light-quantum.

A collision between two light-quanta of energies $h\nu$ and $h\nu'$ respectively may be treated very simply with the aid of the principles of the conservation of energy and momentum.

Assuming that the quanta combine to form a field-molecule of mass $m$ moving with velocity $v$, we have the equations

$$\left.\begin{aligned} h\nu + h\nu' &= mc^2\beta, \\ h\nu\, l + h\nu'\, l' &= mc\beta v_x, \\ h\nu\, m + h\nu'\, m' &= mc\beta v_y, \\ h\nu\, n + h\nu'\, n' &= mc\beta v_z, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

where

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and $(l, m, n)$, $(l', m', n')$ are the direction cosines of the lines of motion of the quanta. Drawing lines OA . OB of length $c$ in the direction of these two lines of motion, the velocity $v$ is represented by OV, where V divides AB so that

$$\nu \,.\, \mathrm{VA} = \nu' \,\mathrm{BV}.$$

Similar equations hold when a field-molecule breaks up into two light-quanta.

It should be remarked that collisions between light-quanta which result in the formation of an exceedingly short-lived

field-molecule are quite compatible with the permanence of cavity radiation, for the existence of the field-molecules may then be ignored. A collision between two light-quanta may, in fact, be supposed to result simply in changes in direction and energy of the quanta that conserve energy and momentum.

Formula (1) indicates that the number of $s$-fold quanta whose frequencies belong to the interval $(\nu, \nu+d\nu)$ and whose directions belong to the solid angle $d\omega$ is

$$\frac{2\nu^2}{sc^3} e^{-\frac{sh\nu}{RT}} d\nu\, d\omega\,;$$

hence the number of such quanta whose momenta belong to the region $dG_x\, dG_y\, dG_z$ is

$$N = \frac{2}{s^4h^3} e^{-\frac{U}{RT}} dG_x\, dG_y\, dG_z,$$

where $U=sh\nu$ is the energy of an $s$-fold quantum of frequency $\nu$.

Now the chance that a definite $s$-fold quantum, which belongs to the momentum region $dG=dG_x\, dG_y\, dG_z$, may hit an $l$-fold quantum $lh\mu$ of frequency $\mu$ belonging to the region of momentum $d\Gamma$ in the interval of time $dt$ and produce a deflexion which, in the system of normal coordinates of Pauli is represented by a line contained in the solid angle $d\Omega_0$, is

$$NK\, u_\mu{}^l\, d\Gamma\, d\Omega_0\, dt = \frac{2}{s^4h^3} Ke^{-\frac{U}{RT}} u_\mu{}^l\, dG\, d\Gamma\, d\Omega_0\, dt,$$

where K is a factor which, according to the investigation of Pauli, is of the form

$$K = \frac{\chi}{E^4U},$$

where $E=lh\mu$ and $\chi$ is invariant under a Lorentz transformation. This quantity $\chi$ depends on the states of the initial and final quanta.

Let the final quanta be $s_1h\nu_1$ and $l_1h\mu_1$ of energies $U_1$, $E_1$ respectively, and suppose that they belong to the regions of momentum $dG_1$, $d\Gamma_1$ respectively, then we have Pauli's relation

$$\frac{dG\, d\Gamma}{UE} = \frac{dG_1\, d\Gamma_1}{U_1E_1}.$$

In equilibrium the direct process will occur just as often as the inverse. The condition for this is

$$\frac{2}{s^4h^3}\mathrm{K}e^{-\frac{\mathrm{U}}{\mathrm{RT}}}u_\mu{}^l\,d\mathrm{G}\,d\Gamma\,d\Omega_0=\frac{2}{s_1{}^4h^3}\mathrm{K}_1e^{-\frac{\mathrm{U}}{\mathrm{RT}}}u_{\mu_1}{}^{l_1}\,d\mathrm{G}_1\,d\Gamma_1\,d\Omega_0.$$

Substituting the values

$$u_\mu{}^l=\frac{8\pi h\mu^3}{c^3}e^{-\frac{lh\mu}{\mathrm{RT}}},\qquad u_{\mu_1}{}^{l_1}=\frac{8\pi h\mu_1{}^3}{c^3}e^{-\frac{l_1h\mu_1}{\mathrm{RT}}},$$

and making use of the relation

$$\mathrm{U}+lh\mu=\mathrm{U}_1+l_1h\mu_1,$$

we find that

$$\frac{\chi}{s^4l^3}=\frac{\chi_1}{s_1{}^4l_1{}^3}.$$

This indicates that the ratio $\chi:\chi_1$ is independent of T, and so equilibrium is possible.

Equations (11) are very similar to some equations which govern the directions of projection of electric dipoles projected from a primary singular point moving with velocity $v$, the radiant field being specified by simple expressions * when the directions of projection are related in this way.

The interpretation which was subsequently given to the equations just mentioned is as follows:—

Consider two particles P and Q moving with velocities less than $c$ along two curves, C and Γ, which are very close together.

A particle travelling with the velocity of light runs backwards and forwards from P to Q. Let $\mathrm{P_1Q_1P_2Q_2}\ldots$ be the zigzag path of this particle, $v$ the velocity of P; then the direction cosines $(l, m, n)$, $(l', m', n')$ of the lines $\mathrm{P_1Q_1}$, $\mathrm{Q_1P_2}$ are connected approximately by the equations

$$\left.\begin{aligned}c\tau\,l+c\tau'\,l'&=v_x(\tau+\tau'),\\ c\tau\,m+c\tau'm'&=v_y(\tau+\tau'),\\ c\tau\,n+c\tau'n'&=v_z(\tau+\tau'),\end{aligned}\right\}\quad\ldots\quad(12)$$

where $\tau$ is the time from $\mathrm{P_1}$ to $\mathrm{Q_1}$, and $\tau'$ the time from $\mathrm{Q_1}$ to $\mathrm{P_2}$. These equations are the same as (11) if

$$\nu\tau'=\nu'\tau.$$

* See the expressions for the potentials in Phil. Mag. Oct. 1913, p. 581, which hold when the directions of projection are related in the present way, and not only in the particular way assumed then.

This means that the frequency $\nu'$ is inversely proportional to $\tau$, and the frequency $\nu$ to $\tau'$. Regarding $\nu$ as approximately the frequency of a field-atom emitted in the direction $P_2Q_2$, we notice that this frequency is inversely proportional to the length of the interval of time between the successive collisions at $Q_1$ and $P_2$.

The equations (12) also occur in the theory of the lines of electric force of a moving electric pole when a line of force is supposed to be made up of particles travelling along straight lines with velocity $c$ and emitted in directions whose direction cosines $(l, m, n)$ vary with the velocity $v$ of the pole in a manner specified by the equations

$$(c^2-v^2)\frac{dl}{dt} = (v_x-cl)\left(l\frac{dv_x}{dt}+m\frac{dv_y}{dt}+n\frac{dv_z}{dt}\right) + \frac{dv_x}{dt}(c-lv_x-mv_y-nv_z) \quad . \quad . \quad . \quad . \quad (13)$$

. . . . . .

Murnaghan has shown * that if $(l, m, n)$ satisfy these equations, then $(l', m', n)$ also satisfy them, $l, m, n, l', m', n'$ being connected by equations of type (12).

## § 5. *Generalized Parallelism and the Equations of Motion of an Electron.*

If some generalization of the theory of § 2 is correct, the equations of motion of an electron need modification, and so it seems worth while to consider some possibilities, using Einstein's theory of gravitation as a guide.

The idea of the path of a particle as a geodesic in a Riemann space is closely connected with Levi Civita's idea of motion by parallelism †, a geodesic being a curve whose tangents are all parallel in the sense of Levi Civita. The idea of a geodesic may be generalized, then, by generalizing the idea of parallelism.

One generalization has already been given by J. Lipka ‡, who has introduced the idea of conformal parallelism. Another generalization, which will be more useful for our purpose, is suggested by the equations (13), or rather the four-dimensional form of these equations.

Using the notation of Einstein's theory, a contravariant

* Amer. Journ. Math. April 1917.

† *Rend. Palermo,* xlii. p. 1 (1917).

‡ 'Publications of the Massachusetts Institute of Technology,' II. June 1923.

vector $\xi^{(n)}$ will be said to move parallel to itself along a curve C whose element of interval is $ds$ if the components of the vector vary along the curve in a manner specified by the differential equations

$$\frac{d\xi^{(n)}}{ds} + \left\{ \begin{matrix} lm \\ n \end{matrix} \right\} \frac{dx^{(l)}}{ds} \xi^{(m)} + g^{(nr)} p_r + g^{(nr)} a_{rm} \xi^{(m)} - \xi^{(n)} p_r \xi^{(r)} = 0,$$

where $p_r$ is a covariant vector, $g_{mn}$ a symmetric and $a_{rm}$ an antisymmetric covariant tensor, and $\left\{ \begin{matrix} lm \\ n \end{matrix} \right\}$ is the Christoffel symbol derived from $g_{mn}$. The quantities $g_{mn}$, $a_{rm}$, and $p_r$ may be supposed to specify the gravitational field and the electromagnetic vector and scalar fields. The contravariant tensor $g^{(nr)}$ is supposed to be derived from $g_{mn}$ in the usual way.

These equations are compatible with the equation

$$g_{mn} \xi^{(m)} \xi^{(n)} = 1 ;$$

and so we may apply them in particular to the contravariant vector

$$\xi^{(n)} = \frac{dx^{(n)}}{ds}$$

if $ds$ is defined by means of the equation

$$ds^2 = g_{mn} dx^{(m)} dx^{(n)}.$$

The resulting equations *,

$$\frac{d^2x^{(n)}}{ds^2} + \left\{ \begin{matrix} lm \\ n \end{matrix} \right\} \frac{dx^{(l)}}{ds} \frac{dx^{(m)}}{ds} + g^{(nr)} p_r + g^{(nr)} a_{rm} \frac{dx^{(m)}}{ds} - \frac{dx^{(n)}}{ds} p_r \frac{dx^{(r)}}{ds} = 0,$$

will be studied as possible equations of motion of an electron ; and to simplify the discussion, we shall consider only the simple case in which

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

In order to obtain a definite case for discussion, we shall assume that $p_n$ is the gradient of a scalar invariant $\Psi$, and

* These equations resemble in some respects the equations of motion considered by L. P. Eisenhart, Annals of Math. xxiv. p. 367 (1923).

shall write our equations of motion in the form :

$$\frac{d}{dt}\left(\frac{mv_x}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\frac{1}{3}e\left[\mathrm{E}_x+\frac{1}{c}(v_y\mathrm{H}_z-v_z\mathrm{H}_y)\right]$$
$$-\frac{2}{3}e\sqrt{1-\frac{v^2}{c^2}}\frac{\partial\Psi}{\partial x}-\frac{2}{3}\frac{e}{c^2}\frac{v_x}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial\Psi}{\partial t},$$

. . . . . .

$$\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\frac{1}{3}e\left[v_x\mathrm{E}_x+v_y\mathrm{E}_y+v_z\mathrm{E}_z\right]$$
$$+\frac{2}{3}e\sqrt{1-\frac{v^2}{c^2}}\frac{\partial\Psi}{\partial t}-\frac{2}{3}\frac{e}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\Psi}{dt},$$

where $m$ is the stationary mass, $e$ the charge, and $(v_x, v_y, v_z)$ the component velocities of the electron, while

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z},$$
$$v^2=v_x^2+v_y^2+v_z^2.$$

These equations have been chosen because they were suggested by, but not rigorously derived from, the electromagnetic theory of § 2. They evidently need some modification because $e\sqrt{1-\frac{v^2}{c^2}}\frac{\partial\Psi}{\partial x}$ is not quite the same as $e\left[\mathrm{E}_x+\frac{1}{c}(v_y\mathrm{H}_z-v_z\mathrm{H}_y)\right]$, although in simple cases the difference between the two is negligible. The equations have been thought to be worth discussing because in the most important case they are readily integrable.

Assuming that the electron moves in the field of a stationary nucleus of charge $-e$, we write

$$\mathrm{E}=-\nabla\Psi,\quad \Psi=-\frac{e}{r},\quad \mathrm{H}=0.$$

Transforming to polar coordinates, we have the equation

$$\frac{m}{\sqrt{1-\frac{v^2}{c^2}}}(r\ddot{\theta}+2\dot{r}\dot{\theta})+\frac{mv\dot{v}r\dot{\theta}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}=-\frac{2e^2}{3c^2}\frac{\dot{r}\dot{\theta}}{r\sqrt{1-\frac{v^2}{c^2}}}$$

for the variation of angular momentum. Writing this

equation in the form

$$m\frac{d}{du}\left(\frac{r^2\dot{\theta}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\frac{2e^2}{3c^2}\frac{r^2\dot{\theta}}{\sqrt{1-\frac{v^2}{c^2}}},$$

where $u=\frac{1}{r}$, we have the first integral

$$\frac{r^2\dot{\theta}}{\sqrt{1-\frac{v^2}{c^2}}}=Ac\,\epsilon^{au}, \qquad a=\frac{2e^2}{3c^2m}, \quad . \quad . \quad (14)$$

where A is an arbitrary constant.

A second integral may be obtained from the energy equation

$$\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right)=-\frac{e^2\dot{r}}{3r^2}\left(1+\frac{2}{\sqrt{1-\frac{v^2}{c^2}}}\right),$$

and we find that

$$1+\frac{2}{\sqrt{1-\frac{v^2}{c^2}}}=B\epsilon^{au},$$

where B is a second arbitrary constant.

Writing

$$v^2=\dot{r}^2+r^2\dot{\theta}^2,$$

and solving for $\frac{d\theta}{du}$, we find eventually that

$$\theta=2A\int\frac{\epsilon^{au}\,du}{\sqrt{(B\epsilon^{au}-1)^2-4(1+A^2u^2\epsilon^{2au})}}.$$

When A and B are chosen arbitrarily, the equation in $u$,

$$(B\epsilon^{au}-1)^2-4(1+A^2u^2\epsilon^{2au})=0,$$

has its smallest positive root as a simple root, and the orbit has an apse at the corresponding value of $u$; but when the equation has two equal positive roots, the orbit approaches a circle asymptotically. Thus an electron can gradually settle down into a circular orbit, or it can gradually get out *, but of course it will take an infinite time for it to get out. The present theory does not give any indication of quantum conditions, but in the author's opinion these may be connected in some way with the equilibrium of the electronic structure in an orbital motion.

* I am indebted to Mr. R. M. Langer for the remark that it can get out.

## II. *Theory of the Condenser in a New Electrostatics.* By H. Bateman (*California Institute of Technology*) *.

### § 1. *The Distribution of Stress in an Electrostatic Field.*

It has been shown recently that, by adopting a new tensor for the specification of the distribution of stress, momentum, and energy in the field of an electron in uniform rectilinear motion, it is possible to account for the existence of discrete electrons with a volume distribution of electricity of one sign, the electricity being everywhere in equilibrium.

In the electrostatic case, when the electron is at rest relative to the axes of coordinates, the tensor takes a simpler form because the retarded potential $\psi$, which is an invariant for the transformations of the restricted theory of relativity, then becomes equal to the electrostatic potential $\phi$. In this case we have in the usual notation †,

$$\left.\begin{aligned} X_x &= \tfrac{3}{4}\left(\frac{\partial\phi}{\partial x}\right)^2 - \tfrac{1}{4}\phi\frac{\partial^2\phi}{\partial x^2} - \tfrac{1}{4}R^2 + 1\tfrac{5}{2}\phi\rho - p, \\ X_y &= \tfrac{3}{4}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y} - \tfrac{1}{4}\phi\frac{\partial^2\phi}{\partial x\partial y} = Y_x, \end{aligned}\right\} \quad . \quad . \quad (1)$$

where $\rho$ is the density of electricity and $p$ a constant which is zero in the æther and different from zero inside the electric charges. The quantity $R^2$ is defined by

$$R^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2,$$

and the associated density of energy is

$$W = \tfrac{1}{12}R^2 + \tfrac{5}{12}\phi\rho.$$

When $\phi = \dfrac{1}{r}$ the components of stress in the æther vanish completely, and so in the present theory the space outside a stationary spherical electron is entirely free from stress. If $\phi$ is a solution of Laplace's equation we have

$$\text{div. } X = 0, \quad \text{div. } Y = 0, \quad \text{div. } Z = 0,$$

and so the stress distribution for the æther seems likely to be a possible one for an electrostatic field of a more general character. In endeavouring to apply this stress-distribution to problems of electrostatics we must remember that in these

* Communicated by the Author.

† The tensor adopted here is that suggested at the top of p. 126 of my paper, "Electromagnetism and Dynamics," Mess. of Math. lii. (1922), except that each component has been divided by three, and a constant mass-tensor is used in place of the variable one.

problems the electricity is not free but is associated with matter, consequently the stress-distribution in regions of space occupied by electricity will not generally be that specified by (1), for the matter exerts forces on the electricity and will give rise to an additional distribution of stress in the electric layer.

## § 2. *The Electrified Soap Bubble.*

In a region occupied by electricity the equations (1) give

$$\text{div. } \mathrm{X} = -\tfrac{1}{3}\rho\frac{\partial\phi}{\partial x} + \tfrac{2}{3}\phi\frac{\partial\rho}{\partial x} = -\rho\frac{\partial\phi}{\partial x} + \tfrac{2}{3}\frac{\partial}{\partial x}(\rho\phi)$$

. . . . . .

There is thus an unbalanced body force $-\rho\nabla\phi + \frac{2}{3}\nabla(\rho\phi)$ which must be cancelled out by a body force exerted by the matter on the electricity. This new body force may be derived from the system of stresses (in the charged space)

$$\mathrm{X}_x' = -\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{2}\mathrm{R}^2 - \tfrac{2}{3}\rho\phi + \mathrm{C},$$

$$\mathrm{X}_y' = -\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y} = \mathrm{Y}_x',$$

. . . . . .

where C is a suitable constant.

Adding the two distributions we obtain the stresses

$$\mathrm{X}_x^* = -\tfrac{1}{4}\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{4}\mathrm{R}^2 - \tfrac{1}{4}\phi\frac{\partial^2\phi}{\partial x^2} - \tfrac{1}{4}\rho\phi + \mathrm{C} - p,$$

$$\mathrm{X}_y^* = -\tfrac{1}{4}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y} - \tfrac{1}{4}\phi\frac{\partial^2\phi}{\partial x\partial y} = \mathrm{Y}_x^*,$$

. . . . . .

and these should furnish the correct value for the stress between the body of a conductor and the outer layer containing the electric charge.

In the case of an electrified spherical soap-bubble we may write

$$\phi = f(r)$$

and consider the stress $\mathrm{X}_x^*$ at a point where $y = z = 0$. Since

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{r}f' + \frac{x^2}{r^2}f'' - \frac{x^2}{r^2}f'$$

$$= f'' \text{ when } x = r,$$

we find that

$$\mathrm{X}_x^* = \frac{1}{2r^4}ff' + \mathrm{C} - p.$$

Now $X_x^*$ should be continuous with the æthereal $X_x$ at the outer spherical boundary $r = a$ and should consequently vanish

$$0 = \frac{1}{2a} f(a) f'(a) + C - p,$$

$$X_x^* = \frac{1}{2r} ff' - \frac{1}{2a} f(a) f'(a).$$

At the inner spherical boundary of the electrified layer the electric field must vanish, consequently $f'(r) = 0$, and so at this boundary the stress $X_x^*$ is

$$X_x^* = -\frac{1}{2a} f(a) f'(a).$$

Now $f(a) = \frac{e}{a}$, $f'(a) = -\frac{e}{a^2}$, where $e$ is the total charge: hence the stress at this boundary has the usual value

$$X_x^* = \frac{e^2}{2a^4}.$$

It must be emphasized that the electrical layer is supposed to contain matter as well as electricity. An attempt to find a distribution of electricity in a spherical shell which will make the body force, $-\frac{1}{3}\rho\nabla\phi + \frac{2}{3}\phi\nabla\rho$, vanish and at the same time give no field at the internal boundary has proved unsuccessful.

## § 3. *The Parallel Plate Condenser.*

In a uniform electric field whose lines of force are parallel to $Ox$ we may write

$$\phi = A + Fx,$$

where A and F are constants. At points in the æther we then have

$$X_x = \tfrac{1}{2}F^2, \quad Y_y = -\tfrac{1}{2}F^2, \quad Z_z = -\tfrac{1}{2}F^2,$$

$$Y_z = Z_x = X_y = 0.$$

Inside the electrical layer we have simply (if $\phi = f(x)$)

$$X_x^* = C - p.$$

The stress component is thus constant and, being continuous with $X_x$ at the boundary, must be equal to $\frac{1}{2}F^2$. This again agrees with the usual theory *.

* It seems almost certain that we shall always obtain results agreeing with the usual theory if $\rho\phi$ is constant over each boundary of a layer of electricity.

It should be noticed that in this case we can find a volume distribution of electricity which will make the body force vanish, and so a layer consisting only of free electric charges seems to be theoretically possible.

The body force is zero, in fact, if

$$-\rho\frac{\partial\phi}{\partial x}+2\phi\frac{\partial\rho}{\partial x}=0,$$

*i. e.* if $\phi=b\rho^2$, where $b$ is a constant. Since $\rho=-\dfrac{\partial^2\phi}{\partial x^2}$, we have the differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\sqrt{\frac{\phi}{b}}=0$$

$$\therefore\qquad \left(\frac{\partial\phi}{\partial x}\right)^2=k^2-\tfrac{4}{3}\sqrt{\frac{\phi^3}{b}},$$

where $k^2$ is a new constant.

If $\phi$ and $b$ are negative on the negative plate of the condenser we may suppose that $\dfrac{\partial\phi}{\partial x}=0$, $\phi=\phi_0$ for $x\leqq 0$, then

$$k^2=\tfrac{4}{3}\sqrt{\frac{\phi_0{}^3}{b}}.$$

The stress at $x=0$ given by equations (1) is now

$$X_x=-\tfrac{2}{3}\phi_0\frac{\partial^2\phi_0}{\partial x^2}-p=\tfrac{1}{2}k^2-p.$$

Since $\dfrac{\partial\phi}{\partial x}$ and the stress $X_x$ should be continuous at the other boundary $x=a$ of the electrical layer

$$p+\tfrac{2}{3}\left(\phi\frac{\partial^2\phi}{\partial x^2}\right)_{x=a}=0,$$

*i. e.*

$$p=\tfrac{2}{3}\sqrt{\frac{\phi^3}{b}},$$

and the stress at $x=0$ is thus

$$X_x=\tfrac{1}{2}k^2-\tfrac{2}{3}\sqrt{\frac{\phi^3}{b}}=\tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2=\tfrac{1}{2}F^2,$$

which is the same value as before. The well-known formula for the stress can thus be obtained in a number of ways. The fact that it has been obtained in the new theory of electrostatics does not necessarily confirm this theory, for one is left with a suspicion that another theory based on a balance of forces may lead to the same result.

III. *The* CuH *Molecule and its Band Spectrum.* *By* HERBERT BELL, *Lecturer in Physics, University of Manchester* *.

THE copper bands in the region λ 4700 to λ 4000 have recently been analysed by Frerichs in Bonn and Bengtsson in Lund †, and the object of the present paper is to derive information from these regarding the copper atom.

Frerichs has represented the fine structure of the resolved bands by the formula

$$1/\lambda = A + B(m-\tfrac{1}{2}) + C(m-\tfrac{1}{2})^2 + D(m-\tfrac{1}{2})^3 + E(m-\tfrac{1}{2})^4, \quad . \quad (1)$$

and his table summarizing results is here reproduced.

| $n_1$. | $n_2$. | Bands. | A. | B. | C. | D. | E. |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 4280 | 23318·16 | 13·497 | −1·0702 | −0·00190 | 0·000049 |
| 0 | 1 | 4650 | 21451·81 | 13·514 | −0·8193 | −0·00198 | 0·000052 |
| 1 | 0 | 4006 | 24929·07 | 12·973 | −1·3395 | −0·00206 | 0·000050 |
| 1 | 1 | 4328 | 23062·65 | 12·956 | −1·0867 | −0·00187 | 0·000053 |
| 1 | 2 | 4690 | 21270·50 | 12·993 | −0·8304 | −0·00221 | 0·000050 |
| 2 | 1 | 4063 | 24584·18 | 12·420 | −1·3665 | −0·00210 | 0·000021 |
| 2 | 2 | 4380 | 22792·57 | 12·418 | −1·1228 | −0·00214 | 0·000050 |

A striking feature is the constancy of the columns giving D and E. The parameter $m$ has, as usual, integral values only and is zero for the "missing" or faint line of the band centre.

As is well known, these bands can be regarded as representing energy changes in the molecule caused by electric field and impact with neighbouring molecules. The molecule in a simple case may have $m'$ quanta of angular momentum about an axis perpendicular to its axis of symmetry, where $m'$ is not necessarily integral, together with $n$ quanta of energy due to vibration along the axis of symmetry or line of join of nuclei. The differences in energy due to changes in $m'$ and $n$ give rise to what may be called "primary" bands situated for the most part in the infra-red, as in the case of the halogen hydrides discussed in a previous paper ‡.

If, now, there be superimposed upon this changes in the electron configuration within the molecule, then the energy changes will be much greater and "secondary" bands appear

* Communicated by Prof. W. L. Bragg, F.R.S.

† Both articles appear in the same issue, Bd. xx. of the *Zt. für Physik*, and are in many respects identical.

‡ Phil. Mag. xlvii. March 1924.

in the visible, as in the present instance. We shall take $n'$ to represent the number of quanta possessed by the oscillating molecule in the higher configuration and $n$ in the lower. Frerichs has assigned corresponding values $n_1$ and $n_2$ respectively for these states, as indicated in the first two columns of his table for the bands investigated. The correctness of his choice will be discussed below. He finds that the corresponding band centres ($m=0$) are well represented by the formula

$$1/\lambda = 23311{\cdot}15 + 1658{\cdot}81n_1 - 44{\cdot}71n_1^2 - 1903{\cdot}48n_2 + 37{\cdot}18n_2^2. \quad . \quad . \quad . \quad (2)$$

In discussing the halogen hydrides (*loc. cit.*) it was found convenient to use an expression derived by Born and Hückel for the energy W of an oscillating and rotating dumb-bell molecule, viz.:—

$$W^n_{m'} = U_0 + nh\nu_0(1+\sigma m'^2) - n^2h\delta + \tfrac{1}{2}h\gamma m'^2 - \tfrac{1}{2}h\gamma^3 m'^4/\nu_0'^2. \quad . \quad (3)$$

U is the force function along the radius having values, along with its derivatives, at the position of equilibrium when at rest of $U_0$, $U_0'$, $U_0''$, etc., $r_0$ is the distance between the nuclei when at rest, $\mu$ is the reduced mass, and $\nu_0$ is the vibration frequency for small oscillations. The following abbreviations are here used :—

$$2\pi\nu_0 = \sqrt{U_0''/\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\beta = r_0 U_0'''/U_0'', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\gamma = h/4\pi^2\mu r_0^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\delta = \frac{5\mu}{48}(\beta^2 - 3r_0^2 U_0^{iv}/5U_0''), \quad . \quad . \quad . \quad (7)$$

$$\sigma = (\beta+2)\gamma^2/2\nu_0^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We shall assume that similar equations hold for both configurations of the present molecule, but in the higher configurations we shall replace $U_0\ \nu_0\ \beta\ \gamma\ \delta\ \sigma$ by $U_0^{(1)}\ \nu_0'\ \beta'\ \gamma'\ \delta'\ \sigma'$ respectively. On changing from the higher configuration $n'$ to the lower $n$ the rotational quanta $m'$ decrease algebraically by unity or from $m'+1$ to $m'$. The frequency of the emitted radiation is then given by

$$h\nu = W^{n'}_{m_1'+1} - W^n_{m'},$$

and on performing the necessary algebra we have

$$\nu = (U_0^{(1)} - U_0)/h + \tfrac{1}{2}\gamma' + n'(\nu_0' + \nu_0'\sigma') - n'^2\delta' - n\nu_0 + n^2\delta$$
$$+ m'(\gamma' + 2n'\nu_0'\sigma') + m'^2\{\tfrac{1}{2}(\gamma'-\gamma) - 3\gamma'^3/\nu_0'^2 + n'\nu_0'\sigma' - n\nu_0\sigma\}$$
$$- 2m'^3\gamma'^3/\nu_0'^2 - \tfrac{1}{2}m'^4(\gamma'^3/\nu_0'^2 - \gamma^3/\nu_0^2). \quad . \quad . \quad (9)$$

[For the Q branch $\nu=(W^{n'}_{m'}-W^{n}_{m'})h$
$=(U_0^{(1)}-U_0)/h+n'\nu_0'-n\nu_0-(n'^2\delta'-n^2\delta)$
$+\frac{1}{2}m'^2\{\gamma'-\gamma+2(n'\nu_0'\sigma'-n\nu_0\sigma)\}-\frac{1}{2}m'^4(\gamma'^3/\nu_0'^2-\gamma^3/\nu_0^2).]$

*Determination of the Band Centre.*

The parameter $m'$ has not been assumed identical with $m$, and we shall write

$$m'=m-\epsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where $\epsilon$ is a fraction, so that the angular momentum of the molecule is not zero at the centre unless $\epsilon=0$ but changes from $(1-\varepsilon)h/2\pi$ to $-\epsilon h/2\pi$. As (1) shows, Frerichs takes $\epsilon=\frac{1}{2}$ in agreement with Kratzer and other writers, and this value will be justified below.

It is noticed at once that (9) explains the fact, shown up in the table, that the coefficients of $m^3$ and $m^4$ are nearly constant from band to band. The value of $\epsilon$ does not affect this argument materially, since the successive coefficients B, C, D fall off so rapidly—B : C : D = 1 : 0·002 : 0·00005. Frerichs, not having the above formula, remarks that "the coefficients D and E, which are merely correction terms, show no systematic variation from band to band."

Heurlinger divides such bands as these into two series separated by the zero line, the P($m$) series being to the left and the R($m$) series to the right on a wave-number diagram. R($m$) is identical with (9) for increasing values of $m'$ from $m=0$ onwards, while for P($m$), $-m$ is written for $m$ and the remaining lines are thus included. We can then break (9) up into

$$R(m)=\text{const.}+(m-\epsilon)(\gamma'+2n'\nu_0'\sigma')$$
$$+(m-\epsilon)^2(\text{const.}+n'\nu_0'\sigma'-n\nu_0\sigma)+\ldots$$
$$P(m)=\text{const.}+(-m-\epsilon)(\gamma'+2n'\nu_0'\sigma')$$
$$+(m+\epsilon)^2(\text{const.}+n'\nu_0'\sigma'-n\nu_0\sigma)+\ldots$$

where the constants do not involve $m$.

Subtracting, we have

$$R(m)-P(m)$$
$$=2m(\gamma'+2n'\nu_0'\sigma')-4m\epsilon(\text{const.}+n'\nu_0'\sigma'-n\nu_0\sigma).$$

If $\epsilon=0$ this function is independent of $n$, *i. e.* of the quantum number of the lower configuration or end stage. Similarly, if $\epsilon=-\frac{1}{2}$ the function is independent of $n'$ the initial stage. Hence, if this function is formed for the various bands of the system and found to be identical for any set of them, we may

deduce that these sets either have all the same initial stage and $\epsilon=0$, or all the same final stage and $\epsilon=-\frac{1}{2}$.

Similarly,

$$\begin{aligned}&\mathrm{R}(m-1)-\mathrm{P}(m+1)\\&\quad=2m(\gamma'+2n'\nu_0'\sigma')-4m(1+\epsilon)(\text{const.}+n'\nu_0'\sigma'-n\nu_0\sigma),\end{aligned}$$

which is independent of $n$ if $\epsilon=-1$, and of $n'$ if $\epsilon=0$. Both these difference functions $\mathrm{R}(m)-\mathrm{P}(m)$ and $\mathrm{R}(m-1)-\mathrm{P}(m+1)$ were applied by Frerichs to these bands and no identities found. On the other hand, he found that $\mathrm{R}(m)-\mathrm{P}(m+1)$ was identical for the three sets of bands $\lambda\lambda$ 4006, 4280 ; 4063, 4328, 4650 ; 4690, 4380. Similarly, $\mathrm{R}(m)-\mathrm{P}(m-1)$ was identical for the three sets $\lambda\lambda$ 4280, 4650 ; 4006, 4328, 4690 ; 4063, 4380.

Now,

$$\begin{aligned}&\mathrm{R}(m)-\mathrm{P}(m+1)\\&\quad=(2m+1)\{\gamma'+2n'\nu_0'\sigma'+(2\epsilon+1)(\text{const.}+n'\nu_0'\sigma-n\nu_0\sigma)\}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{R}(m)-\mathrm{P}(m-1)\\&\quad=(2m-1)\{\gamma'+2n'\nu_0'\sigma'+(2\epsilon-1)(\text{const.}+n'\nu_0'\sigma'-n\nu_0\sigma)\}.\end{aligned}$$

If $\epsilon=\frac{1}{2}$, the second difference is independent of $n$ while the first one is independent of $n'$. *These bands can therefore be grouped into sets according to their initial and final configurations and at the same time the half quantum* $\epsilon=\frac{1}{2}$ *is justified.* This gives the grouping adopted by both Frerichs and Bengtsson as in the above table.

The band centres on substituting $m'=-\frac{1}{2}$ in (9) are now given by

$$\begin{aligned}\nu=(\mathrm{U}_0^{(1)}-\mathrm{U}_0)/h+\tfrac{1}{8}(\gamma'-\gamma)-\tfrac{1}{2}\gamma'^3/\nu_0'^2-\tfrac{1}{32}(\gamma'^3/\nu_0'^2-\gamma^3/\nu_0^2)\\+n'\nu_0'(1+\sigma'/4)-n'^2\delta'-n\nu_0(1+\sigma/4)+n^2\delta. \quad . \quad (11)\end{aligned}$$

*Application of Formula.*

Comparing equations (1) and (9) we have, since $m-\frac{1}{2}$ is now seen to be $m'$,

$$c\mathrm{B}=\gamma'+2n'\nu_0'\sigma', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$c\mathrm{C}=\tfrac{1}{2}(\gamma'-\gamma)-3\gamma'^3/\nu_0'^2+n'\nu_0'\sigma'+n\nu_0\sigma, \quad . \quad (13)$$

and therefore

$$c(\mathrm{B}-2\mathrm{C})=\gamma+6\gamma'^3/\nu_0'^2+2n\nu_0\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It follows that for series having the same end configuration ($n'$ alone varying) $c\mathrm{B}$ should vary from band to band by the

constant amount $2\nu_0'\sigma'$ and $c$C by the amount $\nu_0'\sigma'$. The following table shows how far this is fulfilled.

| $n_1$. | $n_2$. | λ. | B. | | C. | |
|---|---|---|---|---|---|---|
| 0 | 0 | 4280 | 13·497 | | −1·0702 | |
| | | | | −·524 | | −·269 |
| 1 | 0 | 4006 | 12·973 | | −1·3395 | |
| | | | | ... | | |
| 0 | 1 | 4650 | 13·514 | | −0·8193 | |
| | | | | −·558 | | −·267 |
| 1 | 1 | 4328 | 12·956 | | −1·0867 | |
| | | | | −·536 | | −·280 |
| 2 | 1 | 4063 | 12·420 | | −1·3665 | |
| | | | | ... | | |
| 1 | 2 | 4690 | 12·993 | | −0·8304 | |
| | | | | −·578 | | −·292 |
| 2 | 2 | 4380 | 12·418 | | −1·1228 | |

Frerichs in fact derives from his table the relation

$$B = 13{\cdot}50 - 0{\cdot}54n_1, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

more or less in agreement with this table, and therefore

$$2\nu_0'\sigma' = -0{\cdot}54c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Frerichs gives the further relation

$$C = -1{\cdot}07 - 0{\cdot}27n_1 + 0{\cdot}25n_2, \quad . \quad . \quad (17)$$

from which, on comparing with (13),

$$\nu_0\sigma = -0{\cdot}25c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

*The Radial Quantum Numbers $n'$, $n$.*

We have now satisfied ourselves that the columns in Frerichs' table assigning integers to $n_1$, $n_2$ can be also assigned to $n'$, $n$ as regards the relative succession of values, but we have no means as yet of determining the absolute values of the latter. The values down the $n$ column for example might all be increased by unity, as in fact Bengtsson has done for both columns. If we knew that the values of $n_1$, $n_2$ were also those of $n'$, $n$ respectively, we could equate coefficients in (2) and (11). Failing evidence, however, we write

$$n' = n_1 + p', \qquad n = n_2 + p,$$

and substitute in (2). Equating coefficients and using the known values of $\nu_0'\sigma'$ and $\nu_0\sigma$, we obtain

$$\delta' = 44{\cdot}71c, \qquad \delta = 37{\cdot}18c, \quad . \quad . \quad . \quad . \quad (19)$$

$$\nu_0'/c = 1658{\cdot}9 + 89{\cdot}4p', \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\nu_0/c = 1903{\cdot}5 + 74{\cdot}4p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

From (1) and (9) we have

$$D = \text{coefficient of } m'^{-3} = -2\gamma'^3/c\nu_0'^2. \quad . \quad (22)$$

Also $$\gamma' - 0{\cdot}54n'c = cB = (13{\cdot}50 - 0{\cdot}54n_1)c$$

by (15), so that

$$\gamma'/c = 13{\cdot}50 + 0{\cdot}54p'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Hence substituting in (22) from (20) and (23) we have, to a first approximation,

$$D = -0{\cdot}00179(1 + 0{\cdot}012p'). \quad . \quad . \quad . \quad (24)$$

This gives Frerichs' values for D for the 4280 band for small values of $p'$, but plainly D would need to be very exactly known to enable us to use (24) for determining $p'$. It does not seem possible therefore to determine $p'$ and $p$ from internal numerical relationships *.

By inspection, however, of (13) it is noticed that since $n'\nu_0'\sigma'$ and $n\nu_0\sigma$ are always small and C always negative,

$$\gamma' < \gamma, \ i.e. \text{ by (6) } r_0' > r_0.$$

Hence the nuclear separation decreases as the body falls back to the lower configuration. The force function, and its derivatives, is therefore very probably much diminished in the higher configuration and $\nu_0' < \nu_0$.

Now by inspection of (20) and (21) it is seen that, if $p'$ exceeded $p$ by more than 2, $\nu_0'$ would exceed $\nu_0$, from which we deduce that $p' - p$ is in any case small. Both Frerichs and Bengtsson take $p' = p$ since this gives identical values for $n'$ and $n$ to the predominant band $\lambda 4280$. This assumption has since received justification from another quarter.

### *The Isotope Lines.*

Frerichs finds for $\lambda 4280$ and $\lambda 4328$ that the lines of higher quantum number $(-m)$ in the P series are accompanied by satellites on the short-wave side. These have since been attributed to the isotope $Cu_{65}$ of copper by Mulliken †, who gives the formula for their distances from the main lines

$$\Delta\nu = (1 - \rho^2)s, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where $s$ denotes the wave-number separation of the line from

* *The constant E.* From the values of $\gamma$ and $\nu_0$ given below we may similarly calculate the coefficient E,

$$E = 0{\cdot}000078 - 0{\cdot}0000054p' + 0{\cdot}0000093p,$$

as against the value 0·000050 found by Frerichs. It must be kept in mind, however, that the coefficients of $m^{-4}$ are practically determined by quite a few lines at the end of the band and that, moreover, a more complete expansion might demand a term in $m^{-5}$ with consequent modification of E.

† 'Nature,' April 5th, 1924.

the band centre, and $\rho$ is the ratio of the "respective molecular vibrations." By (4) $\rho^2=\mu_2/\mu_1$, where

$$1/\mu_2=1/M_2+1/m, \qquad 1/\mu_1=1/M_1+1/m,$$

and assuming, as established below, that $M\geqslant m$ we easily find that

$$1-\rho^2=m(M_1-M_2)/M^2.$$

The following table by Mulliken gives the agreement between observed and calculated values of $\Delta\nu$ for the two bands.

| λ4280 Band. | | | | λ4328 Band. | | | |
|---|---|---|---|---|---|---|---|
| *m.* | *s.* | $\Delta\nu$ calc. | $\Delta\nu$ obs. | *m.* | *s.* | $\Delta\nu$ calc. | $\Delta\nu$ obs. |
| −22 | 804 | 0·39 | 0·31 | −17 | 539 | 0·32 | 0·23 |
| −23 | 861 | 0·42 | 0·29 | −18 | 588 | 0·34 | 0·25 |
| −24 | 920 | 0·45 | 0·45 | −19 | 639 | 0·37 | 0·26 |
| −25 | 981 | 0·47 | 0·44 | −20 | 691 | 0·40 | 0·22 |
| −26 | 1043 | 0·51 | 0·68 | −21 | 746 | 0·42 | 0·45 |
| −27 | 1107 | 0·53 | 0·56 | | | | |
| −28 | 1172 | 0·57 | 0·60 | | | | |

The isotopes are all supposed to have the same arrangements of electrons, and since gravitational attraction plays only a negligible part in the dynamics of the system the quantities $U_1$, $r$ and $\beta$ in equations (4) .... (9) are unaffected. Other quantities are affected, through involving $\mu$. We have

$$\Delta 1/\mu=1/\mu_2-1/\mu_1=(M_1-M_2)/M^2,$$

so that with $M\geqslant m$ as before,

$$\mu\Delta 1/\mu=1-\rho^2.$$

By (6), since $\gamma\propto 1/\mu$,

$$\Delta\gamma/\gamma=\mu\Delta 1/\mu=1-\rho^2.$$

Similarly, since by (4) $\nu_0\propto 1/\mu^{\frac{1}{2}}$,

$$\Delta\nu_0/\nu_0=\tfrac{1}{2}(1-\rho^2).$$

By (7) $$\Delta\delta/\delta=\Delta\gamma/\gamma=1-\rho^2,$$

and by (4) and (8) $$\nu_0\sigma\propto 1/\mu_{3/2},$$

so that $$\Delta\nu_0\sigma=\tfrac{3}{2}(1-\rho^2).$$

Substituting these results in (9) we have

$$\begin{aligned}\Delta\nu/(1-\rho^2)&=(\nu_2-\nu_1)/(1-\rho^2)\\&=\tfrac{1}{2}(n'\nu_0'-n\nu_0)+3n'\nu_0'\sigma'(m'+\tfrac{1}{2})-(n'^2\delta'-n^2\delta)\\&\quad+(m'+\tfrac{1}{2})\gamma'+m'^2\{\tfrac{1}{2}(\gamma'-\gamma)-6\gamma'^3/\nu_0'^2+\tfrac{3}{2}(n'\nu_0'\sigma'-n\nu_0\sigma)\}\\&\qquad-4m'^3\gamma'^3/\nu_0'^2. \quad . \quad (26)\end{aligned}$$

The distance $s$ of the line from the band centre is obtained

by subtracting (11) from (9). With the result thus obtained (26) can to a sufficient approximation be expressed in the form

$$(\nu_2-\nu_1)/(1-\rho^2) = s+\tfrac{1}{2}(n'\nu_0'-n\nu_0)-(n'^2\delta'-n^2\delta)-2m'^3\gamma'^3/\nu_0'^2, \quad (27)$$

which is practically Mulliken's formula for $n'=n=0$. For the case of copper hydride $M_2-M_1=65-63=2$ and $m=1$, so that $1-\rho^2=-0{\cdot}00252$.

Taking $n'=n=1$ the correction to be applied to $s$ by (27) amounts when multiplied by $1-\rho^2$ to $-0{\cdot}06$. With this correction the column $\Delta\nu$ calc. for $\lambda$4328 is obtained. On the other hand, for $n'=n=0$ for $\lambda$4280 (25) applies directly and the corresponding $\Delta\nu$ obtained. A further correction of 0·06 either way to these columns would be a definite misfit, and although the observed and calculated columns are not quite in step, we must conclude that $n'=n=0$ for $\lambda$4280 and that therefore $p'=p=0$. A difference of unity or more in the $n'$ $n$ values for these bands would plainly, since $\nu_0$ and $\nu_0'$ are large, be completely at variance with these results. Bengtsson takes $p'=p=1$ on account of two additional bands $\lambda$4607 and $\lambda$4228, which in the present scheme would require negative values of $n'$ and $n$. The latter band, however, is obtained from earlier less accurate work of Hartley's, and the former, although visible on Bengtsson's plates, could not be analysed and may well belong to some other system.

*Molecular Constants.*

By (6),

$$r_0'^2=\frac{h}{4\pi^2\gamma'\mu}=\frac{h}{4\pi^2\gamma' M m_H}(1+M/X),$$

where M is the mass of the copper nucleus and X that of the other component of the molecule. Writing $M=63m_H$ and $X=xm_H$, where $m_H=1{\cdot}662\times10^{-24}$ gm. is the mass of the hydrogen nucleus, we obtain on substituting for $\gamma'$ from (23), with $h=6{\cdot}55\times10^{-27}$ and $p'=0$,

$$r_0'=0{\cdot}1978\sqrt{1+63/x}\,.\,10^{-8}\text{ cm.}$$

Writing $x=1{\cdot}008$,

we have $r_0'=1{\cdot}578$ ångströms,

whereas $x=16$, corresponding to a CuO molecule or any similar heavy constituent, gives values of $r_0'$ much smaller than is consistent with X-ray data for the radius of the copper atom. We are therefore dealing with a CuH molecule. The probability of this is kept in mind by Frerichs, who points out that the presence of hydrogen in some form is necessary for the production of the bands. Kratzer was

the first * to point out that many of the metallic bands are due to hydrides.

From (14) and (18)

$$\gamma/c = B - 2C - 0{\cdot}006 = 15{\cdot}63,$$

and by (6) $r_0 = 1{\cdot}467 \times 10^{-8}$ cm.

Also $r_0'/r_0 = \sqrt{\gamma/\gamma'} = 1{\cdot}076,$

so that there is roughly an increase of 8 per cent. in linear dimensions in the higher configuration.

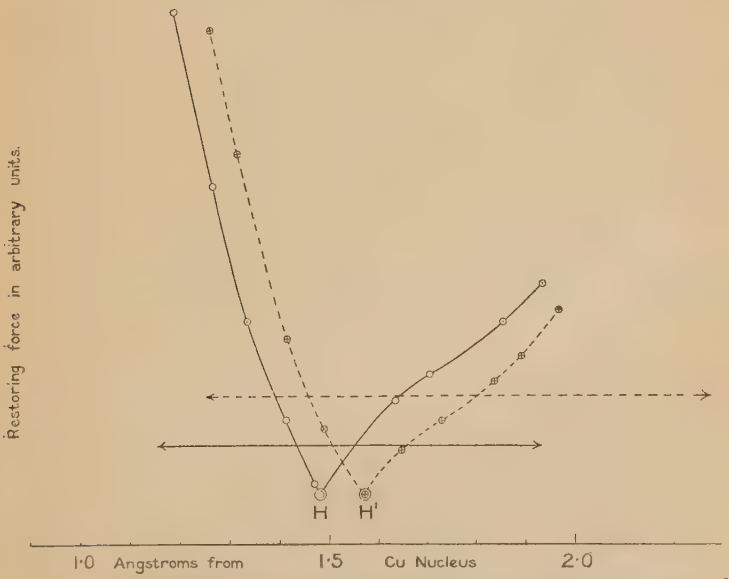
Again, $\nu_0'/c = 1658{\cdot}9, \quad \nu_0/c = 1903{\cdot}5.$

These fundamental frequencies correspond to infra-red primary band systems of wave-lengths $6{\cdot}03\,\mu$ and $5{\cdot}25\,\mu$ respectively. If they or their overtones could be observed, the values of $n'$ and $n$ would be placed beyond doubt.

Solving equations (16), (18), and (8) we readily find

$$\sigma' = -1{\cdot}63 \times 10^{-4}; \quad \sigma = -1{\cdot}32 \times 10^{-4},$$
$$\beta' = -7{\cdot}9 \quad ; \quad \beta = -6{\cdot}9.$$

Fig. 1.




The acceleration of the hydrogen nucleus when displaced a distance $x$ from its equilibrium position is by formula (20) of the previous HCl paper given by

$$\ddot{x} = -4\pi^2\nu_0^2 x\{1 + \tfrac{1}{2}\beta\, x/r_0 + \tfrac{1}{8}(5\beta^2 - 48\,\delta/\gamma)(x/r_0)^2 + \ldots\}.$$

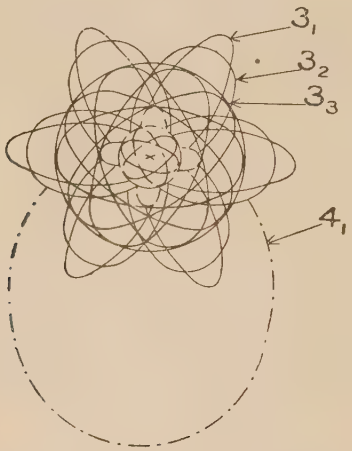
This function is drawn to scale in arbitrary units in fig. 1.

* *Ann. d. Phys.* lxxi. p. 102 (1923).

The full lines represent the lower configuration and the broken ones the higher. The greatest amplitude of oscillation in each case is also indicated by the horizontal lines. The acceleration curves are not drawn near the end of the oscillation range, since the above expansion is not convergent enough to be accurate for high values of $x/r$. The change from lower to higher configuration seems to be accompanied by a uniform expansion of the electric field.

Fig. 2 is a reproduction of the Kramers diagram for the copper atom.

Fig. 2.




The writer wishes to thank Professor W. L. Bragg for the facilities given and for his kindly interest in this work.

*Summary.*

A formula for band spectra in the visible is developed and applied to the copper bands recently measured by Frerichs and Bengtsson.

The half quantum for the rotating molecule is established and the molecule itself verified as being CuH.

The distances between the copper and hydrogen nuclei for the initial and final configurations are calculated and a diagram given showing the variation of the radial force function with distance.

Physics Department,
University of Manchester,
June 1924.

## IV. *The Influence of the Length of the Radiating Column on the Width of Spectral Lines.* By B. Venkatesachar *.

[Plate I.]

In a paper entitled "On the Widening of Spectral Lines" † the late Lord Rayleigh has discussed the five possible causes of the widening of spectral lines.

The Doppler effect arising from the thermal agitation of the luminous centres (the first of the five causes) accounts, almost within the limits of experimental error, for the entire width of a line in the radiation from a thin layer of low-density luminous gas. In the case of a long column, "complications arising from the multiplicity of sources in the line of sight" must operate and increase the width of a line. The radiations corresponding to the less intense parts of a line in passing through the luminous vapour between the place of their origin and the observing optical system may be expected to suffer proportionately smaller absorption than the more intense radiations in the centre of the line, and therefore to be relatively brighter in the light from a long column than in that from a thin layer. A striking instance of the relative increase in brightness of the weaker members of a group of neighbouring lines in the radiation from a long column is provided by the line 5461 Å in the spectrum of the radiation from a low-density mercury-vapour lamp. When the light from a thin layer of the luminous vapour is examined in spectroscopes of high resolving power, this line can be resolved into a group of twelve lines of greatly varying intensity. But in the end-on radiation from a column one metre long carrying a current of not much more than 0·05 amp./sq. cm. all the satellites except −0·24 are found to be nearly of the same brightness ‡. Though an effect of this kind ought to manifest itself in the case of a simple line, there seems to be no instance in which it has been detected or measured, for Lord Rayleigh in the paper alluded to says "it must certainly operate, and yet it does not appear to be important in practice." He alludes in his paper to some experiments carried out by him in which the radiation from a number of sodium flames placed one behind the other was examined. It was found

* Communicated by Prof. A. W. Porter, D.Sc., F.R.S., F.Inst.P

† Phil. Mag. xxix. pp. 274–284 (1915).

† Metcalfe & Venkatesachar, Proc. Roy. Soc. A, cv. pp. 520–531 (1924).

that the presence of the flames behind the one nearest the observing system did not increase the brightness or the width of the $D_1$ and $D_2$ lines. Buisson and Fabry* in an important paper "on the width of spectral lines and the kinetic theory of gases" state that "this cause of widening can intervene in the case of flames of great thickness, but plays no role in the emission by the gas of Geissler tubes." In making reference to the case of flames of great thickness, these authors have possibly in mind Wanner's statement that he observed a broadening of the D lines in the radiation from a sodium flame placed between two concave mirrors. But Wood† has repeated these experiments, and does not confirm Wanner's observation. Hicks in his 'Analysis of Spectra' (1922, p. 14) says that "in practice, however, no effects are observed which can be put down to these causes," one of these causes being a possible damping of the emitted light, and the other the presence of a number of emitting centres in the line of sight. The heavy absorption exerted by luminous mercury vapour on the lines of the first and second subordinate series of triplets, viz. $1\,pi—md$ and $1\,pi—ms$, in the arc spectrum of mercury suggested to the author the possibility of detecting and measuring the widening due to this cause.

*Preliminary Considerations.*

(i.) In the case of a thin layer of low-density luminous gas the only cause of widening is the Doppler effect arising from the thermal agitation of the luminous centres. The distribution of intensity in such a case is governed by Maxwell's law of velocity distribution, and is represented by the relation

$$E = E_0 e^{-k(\lambda-\lambda_0)^2}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $E_0$ is the intensity of the line in the centre and E that at $\lambda$. Also

$$k = \frac{M}{2R\theta} \cdot \frac{c^2}{\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$M$ = mass of luminous centre,
$R$ = the universal gas constant,
$\theta$ = the absolute temperature of the gas,
$c$ = the velocity of light.

* Buisson & Fabry, *Journ. de Phys.* ii. pp. 442–464 (1912).
† Wood's 'Optics,' p. 596.

If wave-length is replaced by wave-number $\nu = \frac{1}{\lambda}$, there will result for the distribution of intensity the relation

$$E = E_0 e^{-k_1(\nu-\nu_0)^2},$$

where $$k_1 = k\lambda^4.$$

(ii.) In an electrically excited gas it is not *a priori* certain that the temperature of the luminous part of the gas is the same as that of the containing tube. But the experimental work of Michelson, and subsequently of Buisson and Fabry, on the visibility of the interference fringes produced by electrically-excited luminous gases, clearly points to the conclusion that the temperature of the luminous part is not appreciably different from the temperature of the tube.

(iii.) The ratio of emission to absorption remains constant within a simple spectral line. In other words, if, in a simple line radiation from a thin layer of low-density luminous gas, E is the intensity of radiation at wave-length $\lambda$ and A the corresponding coefficient of absorption, the ratio $\frac{E}{A}$ remains the same throughout the line. This constancy of $\frac{E}{A}$ throughout a simple spectral line may be shown to result from the following considerations based on the quantum theory. With Einstein, let us consider each individual act of emission or absorption of radiation by an atom as a directed process resulting in a transference of impulse to the atom. Let $E_r$ and $E_a$ represent the internal energies of an atom of the radiating gas in the emitting and absorbing states respectively. $E_r$ and $E_a$ are independent of the translatory motion of the atom. If an excited atom in the energy level E in passing to the energy level $E_a$ emits in the line of sight the quantum $h\nu$ of frequency $\nu$, the resolved part of its velocity in this direction changes as a consequence of the recoil due to radiation from, say, $u_1$ to $u_2$. Considerations of energy and impulse before and after emission give respectively the equations :

$$\tfrac{1}{2}mu_1^2 + E_r = \tfrac{1}{2}mu_2^2 + E_a + h\nu \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

and

$$m(u_1 - u_2) = \frac{h\nu}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

As $u_1$ is small in comparison with $c$ the velocity of light, higher powers of $\frac{u_1}{c}$ than the first do not matter, and the

principles of ordinary mechanics are applicable. Writing $hc\nu_0$ for $E_r - E_a$, we get from (i.) and (ii.) the relation

$$\nu = \frac{\nu_0}{1 - \frac{u_1}{c} + \frac{u_1 - u_2}{2c}}.$$

Since $\frac{u_1 - u_2}{2c}$ is negligible in comparison with $\frac{u_1}{c}$, we may write

$$\nu = \frac{\nu_0}{1 - \frac{u_1}{c}} \quad \text{or} \quad = \nu_0\left(1 + \frac{u_1}{c}\right); \quad . \quad . \quad . \quad \text{(iii.)}$$

and $\nu_0$ may be taken to be the wave-number of the radiation emitted by an atom at rest. Equation (iii.) embodies the ordinary Doppler effect in the case of a source approaching the observer with velocity $u_1$.

Since an elementary act of absorption results in the transition of an atom from the energy level $E_a$ to the energy level $E_r$, the quantum $h\nu$ propagated in the line of sight may be absorbed by an atom in the state $E_a$, only when the line of sight component of its velocity just before absorption is $u_2$; for equations (i.) and (ii.) hold also in the case of absorption, and $E_r$, $E_a$, and $\nu$ remaining unaltered, $u_1$ and $u_2$ do not change. As has been pointed out before, $u_1 - u_2$ is negligible in comparison with $u_1$. In the case of an atom of mercury $u_1 - u_2$, i. e. $\frac{h\nu}{m}$, is about 0·36 cm. sec.$^{-1}$ for a line at 5461 Å. Therefore, in estimating the concentration of the emitters of any given frequency in a simple line radiation and that of the corresponding absorbers according to the Maxwellian law of distribution of velocities, we are justified in taking their respective velocities, viz. $u_1$ and $u_2$, to be equal.

In unit volume of a luminous gas the number of atoms $dN$ in the energy level $E_r$, whose velocities in the line of sight lie between $u$ and $u + du$, is given by

$$dN_r = N_r \sqrt{\frac{\beta}{\pi}} e^{-\beta u^2} du.$$

Similarly, the number of absorbing centres $dN_a$ in the energy level $E_a$ within the same range of velocities is

$$dN_a = N_a \sqrt{\frac{\beta}{\pi}} e^{-\beta u^2} du.$$

$N_r$ and $N_a$ are the respective concentrations of the atoms in the states $E_r$ and $E_a$. If the frequency corresponding to

the line-of-sight velocity $u$ is $cv$, E the intensity of radiation at $\nu$ per unit thickness of a thin layer of unit cross-section is proportional to

$$N_r \sqrt{\frac{\beta}{\pi}} e^{-\beta u^2},$$

and the corresponding coefficient of absorption is proportional to

$$N_a \sqrt{\frac{\beta}{\pi}} e^{-\beta u^2}.$$

The ratio of emission to absorption at $\nu$ is given by

$$\frac{E}{A} = \frac{C_r N_r}{C_a N_a}, \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

and is the same at all points of the line. In ($\alpha$), $C_r$ may be termed the emission factor for the transition $E_r \rightarrow E_a$, and $C_a$ the absorption factor for the transition $E_a \rightarrow E_r$. $C_r$ has a significance analogous to that of the radioactive constant in radioactivity. $C_r$ and $C_a$ are independent of the mode of excitation of the gas, and depend only on the states $E_r$ and $E_a$.

It is possible for two lines to have the same ratio $\frac{N_r}{N_a}$, and yet for $\frac{E}{A}$ to vary from one line to the other on account of the factors $C_r$ and $C_a$ being different for the two. In the case of elements of group II. of the periodic table, emission and absorption lines corresponding to $1S - 1p_1$ and $1S - 1p_2$ have not been detected, and are ruled out according to Sommerfeld by the principle of selection. In these cases $C_r$ and $C_a$ separately vanish. For corresponding lines of isotopes we should expect the ratio of emission to absorption to be the same, for the factors $C_r$ and $C_a$ and the concentration ratio of the emission and absorption centres, viz. $\frac{N_r}{N_a}$, of one line are the same as the corresponding ones of the other.

$\frac{N_r}{N_a}$ may depend on the mode of excitation of the gas. It was found by Metcalfe and the author* that low-density mercury vapour through which a true arc discharge is passing exhibits much higher absorption for lines of the series $1p_i - md$ and $1p_i - ms$ than when the discharge is not a true arc discharge, though the emission line may be actually brighter in the latter case.

* Proc. Roy. Soc. A, c. pp. 149–166 (1921).

To take a particular instance, in a mercury-vapour lamp designed so as to run on low current densities, in which the length of the discharge was 13 cm., there was a sudden and marked increase in the absorption exhibited by the luminous vapour for lines of the series mentioned above, when the voltage between the electrodes fell from 60 to 20 volts. Since the emitting centres are in higher energy levels than the corresponding absorbing centres, the above experimental result implies that the conditions in an arc discharge are favourable for the production of excited atoms of lower energy level in relatively larger numbers than in a non-arc discharge. For pressures of the order of 0·1 mm. of mercury the electric intensity in luminous mercury vapour is least, and is approximately independent of the current density when the radiation is the result of an arc discharge. It appears, therefore, that an increase in the electric intensity has the effect of producing a relatively larger number of excited atoms in higher energy levels. It is of interest to remember in this connexion that a rise of temperature in thermal excitation increases in concentration of atoms in higher energy levels.

## I.

### *Intensity distribution in a simple line radiated from the end of a luminous column of finite length.*

Let AB represent a column of luminous gas contained in a uniform tube of unit cross-section, the observer's spectroscope being situated on its axis at a distance large compared with the length of the column. If $E_0 dx$ is the intensity in the middle of the line radiated by a thickness $dx$ along the axis of the tube, the intensity at wave-length $\lambda$ is given by

$$E dx = E_0 e^{-k(\lambda-\lambda_0)^2} dx.$$

Representing the coefficient of absorption per unit thickness of the column for radiation of wave-length $\lambda$ by A, we write $A dx$ for the coefficient of absorption of a layer of thickness $dx$.

The ratio

$$\frac{E dx}{A dx} = \frac{E_0 e^{-k(\lambda-\lambda_0)^2}}{A}$$

is the same at all points of the line, and will be denoted by I and

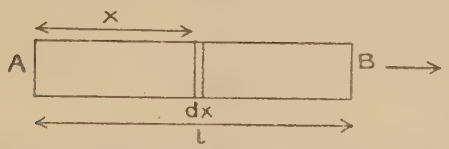
$$A = \frac{E_0}{I} e^{-k(\lambda-\lambda_0)^2} = A_0 e^{-k(\lambda-\lambda_0)^2},$$

where $A_0$ is the coefficient of absorption for radiation of wave-length $\lambda_0$ corresponding to the centre of the line.

The axial radiation $dR$ along AB in the wave-length interval $\lambda$ and $\lambda+d\lambda$ from an element $dx$ at a distance $x$ from the remote end A of the radiating column AB of length $l$ is given by

$$dR=Ee^{-A(l-x)}dxd\lambda.$$

Fig. 1.




So the radiation along AB from the whole column $l$ is given by

$$\int dR=Ed\lambda\int_0^l e^{-A(l-x)}dx$$

$$=\frac{Ed\lambda}{A}(1-e^{-Al})$$

$$=Kd\lambda. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Making the substitutions I and $A_0e^{-k(\lambda-\lambda_0)^2}$ for $\frac{E}{A}$ and A respectively in (3), we get for the intensity distribution in the line radiated by the column the equation

$$K=I\left\{1-e^{-A_0le^{-k(\lambda-\lambda_0)^2}}\right\}. \quad . \quad . \quad . \quad . \quad (4)$$

Writing $K_0$ for the intensity in the centre of the line, *i. e.* at $\lambda-\lambda_0=0$,

$$\frac{K}{K_0}=\frac{1-e^{-A_0le^{-k(\lambda-\lambda_0)^2}}}{1-e^{-A_0l}}=r\,(\text{say}), \quad . \quad (5)$$

which leads to the relation

$$\lambda-\lambda_0=\left\{\frac{\log F}{k}\right\}^{\frac{1}{2}},$$

where

$$F=\frac{A_0l}{\log D}$$

and

$$D=\left\{1-r+re^{-A_0l}\right\}^{-1}. \quad . \quad . \quad . \quad (6)$$

In order to draw the intensity curve, the values of $\lambda - \lambda_0$ are calculated for assigned values of $r$, i. e. $\frac{K}{K_0}$. In the following calculations I have taken the case of a luminous column one metre long. This length was actually used in certain absorption experiments by Metcalfe and the author (*loc. cit.*). 0·4 is taken as the value of $A_0$. This was the experimental value of the absorption coefficient for the complete green-line group 5461 Å in the radiation of low-density mercury vapour carrying a current of 20 milli-amperes per sq. cm., so that 0·4, the coefficient in the centre of a simple line, say the main line of 5461 Å group under the conditions of the experiment alluded to, is, if anything, an underestimate.

In the following table the ratios ($r$) of the intensities at different points in the line to that at the centre are shown against the positions $\lambda - \lambda_0$, the temperature being 400° absolute.

TABLE I.

| $r = \frac{K}{K_0}$. | $\lambda - \lambda_0$. |
|---|---|
| ·99 | ·00488 |
| ·90 | ·00560 |
| ·80 | ·00595 |
| ·70 | ·00621 |
| ·50 | ·00668 |
| ·30 | ·00721 |
| ·20 | ·00756 |
| ·10 | ·00808 |
| ·05 | ·00855 |

The results are shown graphically in fig. 2.

For purposes of comparison, the intensity distribution curve for the radiation from a thin layer is also shown. The wave-length scale is the same for both curves, but the intensity scale in the case of the thin-layer radiation is so chosen as to make the curves touch at $\lambda = \lambda_0$.

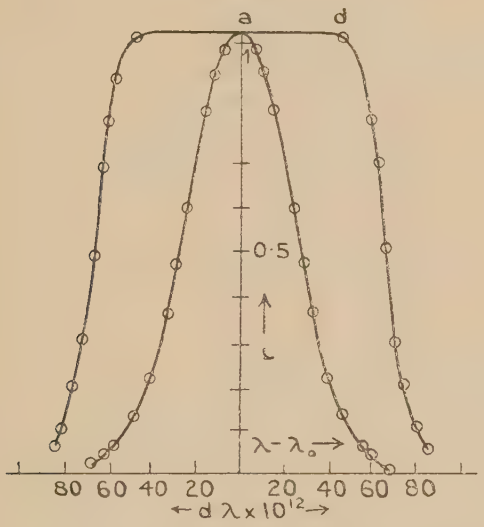
If with Rayleigh we denote the half-width of a symmetrical line as the wave-length separation * between its centre and a point on either side where the intensity falls to one-half the maximum and denote it by $w$, we see that the

* There is some diversity in the conventional definition of the width of a line. According to the above definition the width is $2w$. Buisson and Fabry substitute $2{\cdot}3w$ for $2w$. According to others, the width of a line is the wave-length separation between two points where the intensity is $\frac{1}{e}$th of the maximum, about $2{\cdot}4w$.

half-width of the line radiated by a column of vapour one metre long considered above is more than twice that of the same line radiated by a thin layer under the same conditions of experiment, the calculated half-width in the latter case being 0·0027 Å.

The marked flattening of the curve in the middle of the line has a large influence in reducing the resolvability of two neighbouring lines. Two lines having the same coefficient of absorption, $A_0=0{\cdot}4$, and separated from each other by a wave-length distance of 0·01 Å$=2ad$ (see fig. 2), could be separated in a spectroscope of the necessary resolving

Fig. 2.




power (of the order of 700,000) when the source is a thin layer of the luminous vapour ; on the other hand, the lines would coalesce and not be separated in the axial radiation of the long column considered, however high the resolving power of the spectroscope may be. Of the satellites of the complex mercury line 5461 Å, the two closest to the main line are of wave-lengths +·0086 Å and −·0087 Å. Since the half-width of the main line is about ·0027 Å, a spectroscope whose resolving power is about 700,000 would show the side satellites separate from the main line in the radiation from a thin layer ; whereas, if the radiation is from a long column of the vapour under suitable conditions of excitation, the widened satellites would fuse with the main line, and not be separated

from it in any spectroscope. Further reference will be made to this aspect of the question in describing experiments in which the widening of lines has been observed.

## II.

*The evaluation of the energy content of a line radiated from the end of a column of finite length.*

The radiation in the wave-length range $\lambda$ and $\lambda+d\lambda$ is given by

$$\mathrm{K}d\lambda = \mathrm{I}\left\{1-e^{-\mathrm{A}_0 l e^{-k(\lambda-\lambda_0)^2}}\right\}d\lambda. \quad . \quad . \quad (7)$$

The energy content of the line on expanding the exponential becomes

$$\int \mathrm{K}d\lambda$$

$$= \mathrm{I}\int_{\lambda-\lambda_0=-\infty}^{\lambda-\lambda_0=\infty}\left\{\mathrm{A}_0 l\, e^{-k(\lambda-\lambda_0)^2} - \frac{(\mathrm{A}_0 l)^2}{2!}e^{-2k(\lambda-\lambda_0)^2} + \ldots \text{ad inf.}\right\}d\lambda.$$

Finally, writing $\xi$ for $\mathrm{A}_0 l$, we have

$$\int \mathrm{K}d\lambda$$

$$= \mathrm{I}\sqrt{\frac{\pi}{k}}\left\{\xi - \frac{\xi^2}{2!\sqrt{2}} + \frac{\xi^3}{3!\sqrt{3}} - \ldots \text{ad inf.}\right\}. \quad . \quad . \quad (8)$$

The summation of the series inside the brackets (S), correct to three places of decimals for values of $\xi$ even as low as 4 or 5 requires the use of an inconveniently large number of terms. When $\xi$ is more than 10, the summation becomes hopelessly cumbersome. In Table II. the calculated values of S are shown against the corresponding values of $\xi$.

Table II.

| $\xi$. | S. |
|---|---|
| 1 | ·725 |
| 2 | 1·112 |
| 3 | 1·341 |
| 4 | 1·489 |
| 5 | 1·593 |
| 6 | 1·672 |
| 10 | 1·909 |

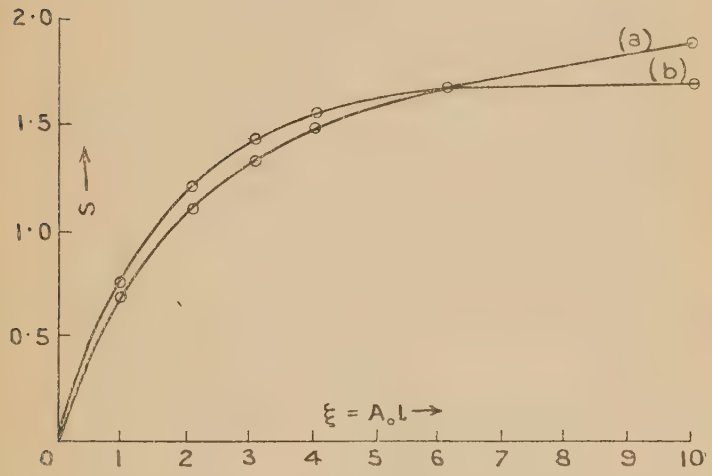
The dominant term in the asymptotic expansion of the

above series S has been determined by G. H. Hardy *, and is

$$2\left\{\frac{\log \xi}{\pi}\right\}^{\frac{1}{2}}$$

This term alone will give the value of S for very large values of $\xi$. The accompanying diagram (fig. 3, $a$) represents the relation between $A_0 l$ and the radiation along the axis (fig. 3). It is plain that the radiation from a luminous column does not asymptotically approach a finite value as the length of the column is increased. Nevertheless, it is manifest that

Fig. 3.




for large values of $\xi$—i. e. $A_0 l$—the rate of increase of S with $\xi$ is small, and approaches zero as $\xi$ approaches infinity; for instance, if we have a luminous column of mercury vapour one metre long, and the current density is so adjusted that the absorption coefficient is ·05, $A_0 l = 5$ and $S = 1{\cdot}593$ (see Table II.). If the column is doubled, other conditions remaining the same, $A_0 l = 10$ and $S = 1{\cdot}909$. Therefore doubling the column does not alter the radiation by more than 0·316, *i. e.* by more than about 20 per cent. If the coefficient of absorption is considered to remain constant throughout the line and equal to 0·56 $A_0$, which is its mean value between the centre of the line and a point where the intensity is 1/10 that at the centre, the radiation R from a

* Proc. of the London Math. Soc. ser. 2, ii. p. 428 (1904).

column of length $l$ is given by

$$R = 1{\cdot}78\,I\left(1 - e^{-0{\cdot}56\,A_0 l}\right)\sqrt{\frac{\pi}{k}}.$$

The radiation asymptotically approaches the value $1{\cdot}78\,I\sqrt{\frac{\pi}{k}}$.

In the case of two neighbouring lines which have the same ratio of emission to absorption but are of unequal intensity in the radiation from a thin layer, the weaker one approaches in brightness the stronger line on increasing the length of the radiating column. If the absorption coefficient is not taken to vary from point to point of a line, the weaker line should more rapidly approach in brightness the brighter one as the length of the radiating column is increased than if the coefficient varied from point to point as postulated in the above discussion. Certain experiments carried out by Metcalfe and the author on the components of the green-line group 5461 Å in the radiation from long columns distinctly support the latter postulate (*loc. cit.*).

*Experimental evidence of the widening of lines due to length of radiating column.*

As the width of a simple line of the 5461 Å group in the axial radiation from a long column of low-density mercury vapour, through which an arc discharge is passing, is of the order of ·014 Å at a temperature of 400° absolute, the corresponding width in the case of the thin-layer radiation being ·006 Å, it becomes necessary to use instruments of high resolving power in experiments carried out with the object of observing the change of width from ·006 Å to ·014 Å when the illuminating source is changed from a thin layer to a long column. In the experiments described below a Lummer-Gehrcke plate, made by Hilger and of resolving power exceeding 200,000 at 5461 Å, was used in conjunction with a constant deviation spectroscope which produced the preliminary dispersion. For a proper understanding of the experiments described below, proving the existence of the widening of lines we have been considering, a knowledge of the performance of the resolving apparatus used is essential. The following are the data of the Lummer and Gehrcke plate I have used in these experiments:—

$$d \text{ (plate thickness)} = 0{\cdot}47 \text{ cm.},$$
$$\mu \text{ at } 5461\text{Å} = 1{\cdot}51108,$$
$$\frac{d\mu}{d\lambda} = -509,$$
$$\delta\lambda = {\cdot}2712 \text{ Å}.$$

The relative positions of the components of the 5461 Å group between two successive orders of the main line are shown in fig. 4, where the full lines stand for the $n$th order components and the dotted ones for those of the $n+1$th order.

The mercury-vapour lamp shown in fig. 5 (i.), was constructed with the object of providing a long low-pressure radiating column GH (length 30 cm. and diameter of tube 0·8 cm.). The horizontal part is the seat of the long column

Fig. 4.

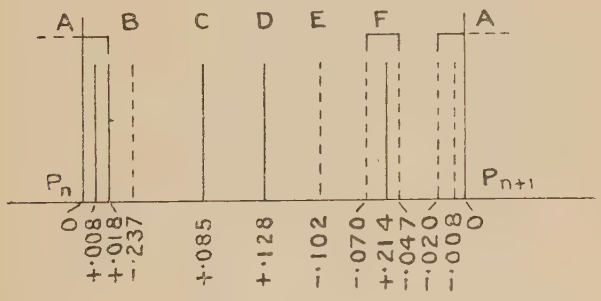
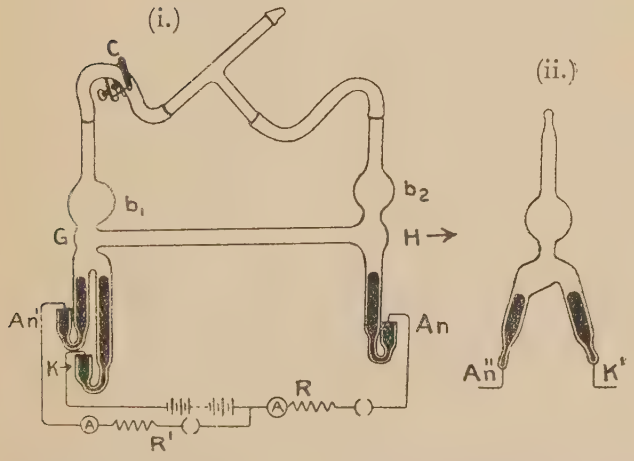



Fig. 5.



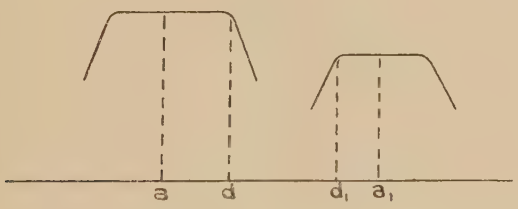

arc discharge between the electrodes An and K. In order to facilitate the starting of this arc, provision is made to run a subsidiary arc between electrodes An′ and K, K functioning as the common cathode for the two arcs. The tube is provided with two bulbs, $b_1$ and $b_2$, which are intended to condense the vapour issuing from the anodes and the

common cathode when the lamp is running. The lamp is evacuated by a mercury pump till the pressure falls below ·01 mm. of mercury. The passage above the subsidiary arc leading to the pump is closed by the screw clip C and the subsidiary arc started. As soon as the stream of mercury vapour issuing from this arc reaches the surface of the anode An, the long arc GH starts, and the resistance R is adjusted so that the current in GH is not less than 1·5 amp. The subsidiary arc is now quenched without affecting the steady running of the main arc. In order to prevent the rush of mercury vapour along GH, connexion with the pump along GC is restored by loosening the clip C. The observing apparatus, consisting of a constant deviation spectroscope and the Lummer-Gehrcke plate already referred to, is placed properly aligned at sufficient distance from the tube to ensure full effectiveness from all parts of the arc GH. When the green-line group 5461 Å is examined in this apparatus, with the resolving plate in position, the difference between the pattern produced by the radiation from the arc just above the anode (thin layer) and that due to the long column radiation is very striking. In addition to being greatly enhanced, all the lines except −·24 are distinctly widened, with the result that the lines of demarcation between the several components become so indistinct that the pattern may be easily mistaken for one produced by a high-pressure lamp. In the reproduction (Pl. I. fig. I) the enlarged pictures of the two patterns are shown in juxtaposition. The appearance presented by −0·24 and +·085, marked B and C, is particularly interesting. In the thin-layer pattern I (*a*), −0·24 is hardly visible, whereas +·085 is bright and sharp, but in pattern I (*b*), produced by the long column, B, −0·24 is greatly enhanced in brightness owing to feeble absorption, and stands out as the brightest and sharpest line of the group, whereas C, +·085, the heavily-absorbed satellite, is comparatively weak but distinctly widened. The difference in sharpness, striking as it is in the reproduction, is much more so when observed in the instrument. The most convincing proof of widening is no doubt to be looked for in the disappearance of the line of demarcation between two close lines as the illuminant is changed from the thin layer to the long column. In fig. 4 are shown the relative positions of the lines in the observed pattern; at $P_{n+1}$ the main line O and the close lines −·008 and −·020 are not resolved in the instrument. The line of separation between groups A and F was very distinct in the thin-layer pattern, but was entirely absent in the pattern due to the long column when the

current in it was of the order of 1·5 amp. This is distinctly shown in the reproduction (see Pl. I. fig. I). Here again the actual appearance in the instrument is much more pronounced and striking. In this case $\xi$ the product of the coefficient of absorption and the length of the radiating column is of the order of 1000 for $-0{\cdot}020$ Å.

The wave-length separation $a\,d$ (see fig. 2) between the centre of the line and the point on one side where the intensity falls to (say) 0·99 of that at the centre increases with $A_0 l$ (see fig. 6). Therefore $a\,d$ for the line $-0{\cdot}020$ Å, which is more heavily absorbed, is greater than $a\,d$ for the line $-{\cdot}047$ Å. If from ·027 the wave-length distance between the lines $-{\cdot}047$ Å and $-{\cdot}020$ Å we subtract $a\,d + a_1\,d_1$ (see fig. 6), which is approximately ·014, we get ·013, and if the instrument is unable to resolve two lines with a separation of ·013 Å, the line of demarcation disappears; and this is so in the case of the instrument used.

Fig. 6.




In this method of observation the instrumental width of a line and its physical width are made to assist each other in wiping out the line of separation between two close lines.

If, however, $a\,d + a_1\,d_1 \geq a a_1$, the two lines fuse together and the line of demarcation disappears, even in an instrument of infinite resolving power, for which the instrumental width of a line is zero. The observed effect is not due to the brightening-up of the group F in a long column. In order to make sure on this point, an experiment was made using as source an arc produced in the tube shown in fig. 5, ii. The pressure was increased to 1 cm. of mercury by the introduction of air, and the arc was run with a current of 2 amp. The group of lines F increased very much in brightness, and the line of demarcation between F and A was distinctly visible. Neither the increase of width due to a pressure of 1 cm. nor the consequent increase of brightness could blur the line of separation; if anything, the line of separation was more distinct than before the increase of pressure and

current. With an *echelon* spectroscope of resolving power 100,000, the line of demarcation between +·085 and +·128, sharp in the pattern produced by the radiation from a thin layer, was distinctly blurred in that of the long-column radiation. The resolving power was reduced by diminishing the aperture of the *echelon* till the line of demarcation between these satellites disappeared in the long-column pattern. Under these conditions the use of a thin luminous layer in place of the long column again revealed the separating line.

The group of lines 5769 Å and 5790 Å were similarly examined, and it was found that the satellites +·044 Å and −·050 Å for 5769 Å were distinctly widened in the long-column radiation, with the result that the lines separating them from the main line, which are sharp in the Lummer-plate pattern, due to the thin-layer radiation, grew indistinct when the long column, GH, was used as the source. These satellites and the main line have been reversed on a continuous background by Metcalfe and the author (*loc. cit.*).

In the case of 5790, the satellites are very much brighter in the long-column radiation, but remain sharp and well separated from each other. This result is in agreement with the fact that it was not possible to reverse them on a continuous background, indicating that their absorption coefficient is of a different order from that of the two satellites of 5769 Å.

Experiments proving the existence of the widening indicated by the theory having been described, the rest of the paper is devoted to a discussion of the bearing of this widening on certain physical investigations in which experimental methods depending on the visibility of interference fringes have been employed. As preliminary to the discussion, the following problem is considered.

## III.

*The visibility of interference fringes produced by the axial radiation from a luminous column of finite length in an interferometer of the Michelson type.*

In the case of a thin layer of gas at a low pressure, in which the emitting centres have a velocity distribution governed by Maxwell's law, the visibility of the interference fringes in an interferometer which produces the pattern by splitting the incident beam into two of equal intensity and after they have travelled different distances subsequently

combining them, has been worked out by the late Lord Rayleigh, and is given by

$$V = e^{\frac{-2\pi^2 R}{c^2}} \left(\frac{2p}{\lambda}\right)^2 \frac{\theta}{M}, \quad . \quad . \quad . \quad . \quad (9)$$

where the visibility V, according to Michelson,

$$= \frac{I\ \text{max.} - I\ \text{min.}}{I\ \text{max.} + I\ \text{min.}} \qquad I = \text{intensity},$$

and $2p$ = the path-difference between the interfering beams.

As a result of photometric experiments, Lord Rayleigh (*loc. cit.*) arrived at the result that the variation of brightness from a maximum to a minimum cannot be recognized when V falls to $\frac{1}{40}$, *i.e.* to ·025.

Setting $\frac{1}{40}$ for V in (9), we get for $\frac{2p}{\lambda}$

$$\frac{2p}{\lambda} = N_0 = \frac{c}{\pi}\sqrt{\frac{\log 40}{2R}}\sqrt{\frac{M}{\theta}}$$

$$= 1{\cdot}427\sqrt{\frac{M}{\theta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The expression for the visibility takes a more complicated form in the case of a column of finite length. If two coherent beams of equal intensity and wave-number $\nu$ overlap, they produce an illumination which is given by

$$2A^2(1 + \cos 4\pi p\nu),$$

where $A^2$ is the intensity of either beam and $2p$ is their path-difference. If the source which has given rise to the two interfering pencils gives a line with an intensity distribution given by (say) $\psi(x)$, where $x$ is the wave-number distance from the centre of the line, the illumination at the point of the interference pattern under consideration is to be evaluated by summing up the illuminations due to the several elements into which the radiation line may be split up. The illumination $dI$ due to the radiations in the wave-number range $x$ and $x + dx$ is given by

$$dI = \psi(x)\{1 + \cos 4\pi p(\nu_0 + x)\}dx, \quad . \quad . \quad (11)$$

where $\nu_0$ is the wave-number for the centre of the symmetrical line. The illumination at the point is obtained by integrating

(11) between the limits $+\infty$ and $-\infty$.

$$\int dI = \int_{-\infty}^{+\infty} \psi(x)\{1+\cos 4\pi p(\nu_0+x)\}dx$$

$$= \int_{-\infty}^{\infty} \psi(x)dx + \cos 4\pi p\nu_0 \int_{-\infty}^{\infty} \psi(x)\cos 4\pi px\,dx$$

$$- \sin 4\pi p\nu_0 \int_{-\infty}^{\infty} \psi(x)\sin 4\pi px\,dx,$$

and $\psi(x)$ by a slight transformation of (6) is given by

$$K d\nu = \psi(x)dx$$

$$= I\left\{1 - e^{-\xi e^{-k_1 x^2}}\right\}dx.$$

As the line is symmetrical, the third integral is zero. Let the first and the second integrals be represented by C and D respectively.

$$C = \int_{-\infty}^{+\infty} \psi(x)dx$$

$$= I\sqrt{\frac{\pi}{k_1}}\left\{\xi - \frac{\xi^2}{2!\sqrt{2}} + \frac{\xi^3}{3!\sqrt{3}} - \frac{\xi^4}{4!\sqrt{4}} + \ldots\right\} \quad (12)$$

The second integral may be written thus

$$D = I\int_{-\infty}^{+\infty}\left[\xi e^{-k_1x^2} - \frac{\xi^2}{2!}e^{-2k_1x^2} + \frac{\xi^3}{3!}e^{-3k_1x^2} - \ldots\right]\cos 4\pi px\,dx.$$

Hence we have

$$D = I\sqrt{\frac{\pi}{k_1}}\left\{\xi e^{-\frac{4\pi^2p^2}{k_1}} - \frac{\xi^2}{2!\sqrt{2}}e^{-\frac{4\pi^2p^2}{2k_1}} + \ldots\right\} \quad (13)$$

The intensity I is given by

$$I = C + D\cos 4\pi p\nu_0.$$

Since the brightness of a line is reduced almost to zero at a wave-number distance $x$ from the centre which is very small compared with $\nu_0$, the variation of intensity in any given region of the interference pattern is governed by $\cos 4\pi p\nu_0$;

*i. e.* a variation in $2p$ which changes the value of $\cos 4\pi p\nu_0$ from $+1$ to $-1$ does not produce any appreciable difference in D. Therefore $I_{max.} = C + D$ and $I_{min.} = C - D$;

$$i.\,e., \qquad V = \frac{I_{max.} - I_{min.}}{I_{max.} + I_{min.}}$$

$$= \frac{D}{C}$$

$$= \frac{\xi e^{-\frac{4\pi^2 p^2}{k_1}} - \frac{\xi^2}{2!\sqrt{2}} e^{-\frac{4\pi^2 p^2}{2k_1}} + \frac{\xi^3}{3!\sqrt{3}} e^{-\frac{4\pi^2 p^2}{3k_1}} - \ldots}{\xi - \frac{\xi^2}{2!\sqrt{2}} + \frac{\xi^2}{3!\sqrt{3}} - \ldots}$$

. . . . . (14)

As the path-difference $2p$ increases, the clearness of the fringes falls, and when $2p$ assumes the value which makes $V = 0{\cdot}025$, they become unrecognizable;

*i. e.*,

$$\frac{\sum_1^\infty (-1)^{n-1} \frac{\xi^n}{n!\sqrt{n}} (\chi)^{\frac{1}{n}}}{\sum_1^\infty (-1)^{n-1} \frac{\xi^n}{n!\sqrt{n}}} = 0{\cdot}025, \quad . \quad . \quad . \quad (15)$$

where $\chi = e^{-\frac{4\pi^2 p^2}{k_1}}$.

It is easily seen that (15) reduces to $e^{-\frac{4\pi^2 p^2}{k_1}} = 0{\cdot}025$ when $A_0 l$ approaches zero, *i. e.* to Rayleigh's equation (10) for the case in which there is no absorption or the radiating layer is thin.

For assigned values of $\xi$ (*i. e.*, $A_0 l$,) $\chi$ may be evaluated by a somewhat laborious method of successive approximation.

TABLE III.

| $\xi = A_0 l$. | $\chi$. |
|---|---|
| 0 | ·025 |
| 1 | ·091 |
| 2 | ·165 |
| 3 | ·233 |
| 4 | ·295 |
| 5 | ·350 |
| 40 | ·565 |

For large values of $\xi$ the method becomes too cumbersome. In this case, however, the visibility can be calculated by considering the graph of the intensity distribution curve to be a trapezium with a flat top (see fig. 2). In the case $A_0 l = 40$ this method has been adopted. The result so obtained is sufficiently accurate for all practical purposes.

Now, since $\chi = e^{-\frac{4\pi^2 p^2}{k_1}}$, we get by suitable transformation after making the substitution $k_1 = k\lambda^4 = \frac{M}{2R\theta} c^2 \lambda^2$,

$$N = \frac{2p}{\lambda} = \frac{c}{\pi}\left\{\frac{-\log \chi}{2R}\right\}^{\frac{1}{2}} \sqrt{\frac{M}{\theta}}$$

$$= \phi(A_0 l)\sqrt{\frac{M}{\theta}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

For a thin layer we have from (10):

$$N_0 = \frac{2p_0}{\lambda} = \frac{c}{\pi}\left\{\frac{\log 40}{2R}\right\}^{\frac{1}{2}} \sqrt{\frac{M}{\theta}} .$$

Therefore

$$\frac{N}{N_0} = \left\{\frac{-\log \chi}{\log 40}\right\}^{\frac{1}{2}} . \quad . \quad . \quad . \quad . \quad (17)$$

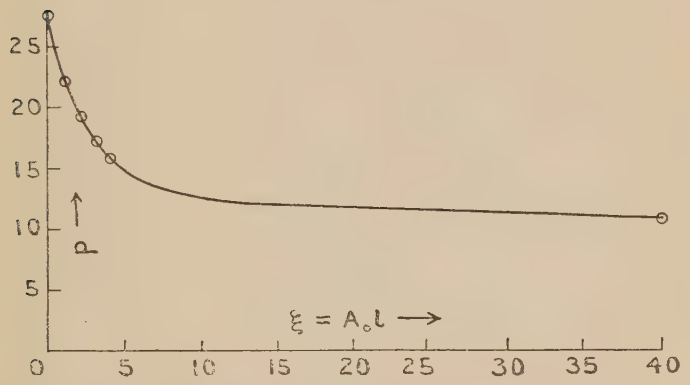
Buisson and Fabry have replaced 1·427 by 1·22 in (10). This means that according to these authors the fringes cease to be recognizable for values of V higher than ·025. In the calculations embodied in this paper, Lord Rayleigh's value 0·025, based on photometric measurements carried out by him, has been used. The use of 0·025 gives higher values for the limiting order of interference at which the fringes cease to be visible, and hence supplies a more severe test of the widening of lines.

TABLE IV.

| $\xi$. | $\chi$. | $\frac{N}{N_0}$. | N. | $p$. |
|---|---|---|---|---|
| 0 | ·025 | 1·000 | $1·01 \times 10^6$ | 27·5 cm. |
| 1 | ·091 | ·805 | $·81 \times 10^6$ | 22·1 ,, |
| 2 | ·165 | ·699 | $·70 \times 10^6$ | 19·2 ,, |
| 3 | ·233 | ·628 | $·63 \times 10^6$ | 17·3 ,, |
| 4 | ·295 | ·573 | $·578 \times 10^6$ | 15·8 ,, |
| 5 | ·350 | ·531 | $·537 \times 10^6$ | 14·7 ,, |
| 40 | ·565 | ·396 | $·40 \times 10^6$ | 10·9 ,, |

According to this, the difference of path $2p$ when the fringes become unrecognizable should fall from $2 \times 27{\cdot}5$ cm. to $2 \times 17{\cdot}3$ cm. when the thickness of the radiating column is increased from 0 to 7 cm. in the case of the arc discharge of mercury vapour considered for which the coefficient of absorption is 0·4 per cm. ; when the thickness of the layer is increased to one metre, it should fall to $2 \times 11$ cm., *i. e.* to 40 per cent. of the value for a thin layer. If a Fabry and Perot interferometer be used, and if the above result which has been obtained by considering only two coherent rays of equal intensity be taken to be applicable to this case also, the fringes should become invisible when the distance between the plates is 11 cm., but with a thin luminous layer under the same conditions of excitation as source they should be recognizable even when the distance is as much as 27·5 cm.

Fig. 7.




The experimental verification of this result in the case of mercury is rendered difficult by the complexities of the heavily-absorbed lines of the first and second subordinate series. Cadmium lines which are comparatively simple would be more suitable for this purpose.

The curve in fig. 7 relating $p$ with $\xi (A_0 l)$ shows that the distance between the plates, $p$, at which the fringes cease to be visible diminishes rapidly with increase of $\xi$ for points on the curve for which $\xi$ is small, and as we move to the right, *i. e.* when $\xi$ is large, this rate of diminution becomes very low. This means that, if the coefficient of absorption is large, a few centimetres of the radiating column are enough to cause considerable falling-off of the limiting order of interference at which the fringes cease to be discernible. The

above considerations suggest that in experiments carried out with the object of determining the masses of radiating particles by visibility experiments, such as those of Buisson and Fabry, the use of an end-on tube should be avoided as likely to give too low a value for the mass of the luminous centres if the line radiation used for the purpose is appreciably absorbed by the luminous vapour.

Experimental values for the limit of interference $\frac{2p}{N} = N$ are slightly lower than the calculated ones, and the discrepancy is explained on the supposition that the temperature of the gas is slightly higher than that of the containing tube. Where end-on tubes have been used, part of this discrepancy at any rate may be attributed to the cause we have been discussing. Knowing the mass of the luminous centre and the limit of interference N, we can calculate from equation (10) the temperature of a thin layer of radiating gas. If the same equation is used to calculate the temperature of a long column, the calculated temperature may be far in excess of the true temperature, as equation (10) takes no account of the widening due to the length of the column. Taking the second case in Table IV. for $\xi = 1$, a radiating column only 2·5 cm. long whose coefficient of absorption is 0·4 per cm. would give an experimental value of 612° absolute for the temperature of the vapour, whereas a thin layer would only give 400° absolute. If we take the sixth case, where the length of the radiating column is increased to 100 cm. under the same conditions of electrical excitation, the fictitious temperature works out to be 2540°.

## IV.

A very interesting application of interference methods has been made by Buisson, Fabry, and Bourget* to the determination of the temperature of the Orion nebula and the masses of the luminous centres which give the lines attributed to nebulium, the hypothetical element supposed to be present in nebulæ. The method consists in producing Fabry and Perot interference-fringes on the image of the nebula in a telescope by means of a suitably placed Fabry and Perot etalon, and taking a series of photographs of the nebula with gradually increasing distances between the plates of the interferometer, the radiation used being the hydrogen line $H_\gamma$. This limit of interference and that found for the same line in the laboratory are used in determining the temperature

* The Astrophysical Journal, xl. pp. 241–258.

of the nebula. Representing by $N_l$ and $N_n$ the limits for the radiations from a laboratory source and the nebula respectively, and by $\theta_l$ and $\theta_n$ the corresponding temperatures, we have

$$N_l = \alpha\sqrt{\frac{M}{\theta_l}}, \qquad N_n = \alpha\sqrt{\frac{M}{\theta_n}};$$

$$\therefore\ \theta_n = \left(\frac{N_l}{N_n}\right)^2\theta_l.$$

Here it is taken that the distribution of intensity in the radiated line in both cases is that of the radiation from a thin layer and of the form $e^{-k(\lambda-\lambda_0)^2}$. Even though the gaseous material of the nebula is at a very low pressure, the widening due to absorption owing to the enormous thickness of the radiating matter may be considerable. In the case of the hydrogen line of the Balmer series, an absorption has been detected and measured by several observers. Jungjohan * finds that in one of his experiments the absorption in the case of $H_\alpha$ and $H_\beta$ was 100 per cent. with a layer of 6 cm. An extreme instance of the enormous influence which this widening may have on the observed temperature is provided by the mercury-arc discharge considered (see Table IV.). When $A_0 l = 3$, *i. e.* when the length of the radiating column is 7·5 cm., the limit of interference under the conditions of excitation cited above is $\cdot 63 \times 10^6$, whereas for a thin layer it is $1\cdot 01 \times 10^6$. Using this tube end on would give 1120° absolute for the temperature of the vapour, whereas the correct temperature as judged by the limit of interference obtained with a thin layer as source of the radiation would be 400° absolute.

A column one metre long, as already stated, would give an estimated temperature of 2540° absolute. In the case of hydrogen the absorption coefficient is small, and probably is considerably less than in the case of the mercury-arc discharge, but enormous thickness of the radiating matter in the case of nebulæ may make up for the possible low value of the absorption coefficient. The uncertainty of the extent of widening due to absorption considerably diminishes the reliance to be placed on determinations of nebular temperatures by interference methods.

Buisson, Fabry, and Bourget remark that 15,000° the temperature of the nebula calculated from their observations, is a maximum, every other cause tending to lower this value

* *Zeitschr. Wiss. Phot.* ix. (Oct., Nov., Dec. 1910)

But the present discussion indicates that the true temperature of the nebula may be far lower than 15,000°. If $A_0 l$ is of the order of 40, the temperature falls down to 2350°. For if $\theta_0$ is the true temperature of the gas and $\theta$ the fictitious temperature calculated without taking into account the effect of absorption in the radiating column, we have from (10)

$$\frac{\theta_0}{\theta} = \left(\frac{N}{N_0}\right)^2,$$

where $N_0$ and N are respectively the limiting orders of interference for the long-column radiation and the radiation from a thin layer of the same gas. If $A_0 l = 40$, $\frac{N}{N_0} = \cdot 396$ (see Table IV., column 3) ; and taking the fictitious temperature to be 15,000°, we have for the true temperature

$$\theta_0 = \cdot 396^2 \times 15{,}000 = 2352° \text{ absolute.}$$

A gas column 1000 kilometres long, with a coefficient of absorption $\cdot 4 \times 10^{-6}$ per cm., is sufficient to give this result. The discovery of dark irregular nebulæ and the resemblance existing between the spectra of bright irregular nebulæ and the spectra of stars involved in them have recently influenced astrophysicists in holding the view that irregular nebulæ are not self-luminous, but that the luminosity is due to the high-temperature stars associated with them. A low temperature for the nebula is consistent with this view. The almost invariable presence of B-type stars near a bright nebulous region and the absence of bright nebulæ near red stars, strongly suggest that the luminosity of the gaseous material of nebulæ, like the Orion nebula, is due to photo-electric excitation and electronic impact produced by high-frequency radiation and electron emission from the high-temperature stars involved in the nebulæ.

What has been said of the temperature determination holds also for the determination of the masses of the luminous centres which give rise to the so-called nebulium lines 3726 Å, 3729 Å, and 5006 Å. Here the data for calculation are supplied by the limiting orders of interference in the case of $H_\gamma$ and the lines of unknown origin. Equation (10) gives the relation

$$\frac{M_x}{M_H} = \left\{\frac{N_x}{N_H}\right\}^2, \quad \ldots \ldots \quad (18)$$

where $M_x$ and $M_H$ are the masses of the luminous centres of

the unknown element and of the hydrogen atom respectively, and $N_x$ and $N_H$ represent the corresponding limiting orders of interference. Equation (1), which takes account of the length of the radiating column, gives

$$\frac{N_x^2}{N_H^2} = \frac{\phi({}_xA_0 l)}{\phi({}_HA_0 l)} \cdot \frac{M_x}{M_H}, \quad . \quad . \quad . \quad . \quad (19)$$

where ${}_xA_0$ and ${}_HA_0$ represent respectively the absorption coefficients of luminous nebulium and hydrogen for the respective lines considered, and $l$ is the length of the radiating column which is taken to be the same for the two lines. If we assume that these coefficients are equal, equations (18) and and (19) become identical. But this assumption is by no means warranted, for it is known that different lines behave differently with respect to absorption; for instance, in the case of a mercury-vapour lamp the coefficient of absorption is large for the lines of the first and second subordinate series of triplets, and comparatively small for other lines. Consequently the limiting order of interference would be much smaller in the case of the heavily-absorbed lines than in the case of the feebly-absorbed ones. If a thin layer is used this discrepancy does not arise, for

$$\operatorname*{Lt}_{A_0 l \to 0} \phi(A_0 l) = 1{\cdot}427.$$

Using equation (18), the authors cited above have come to the conclusion that there are two unknown elements in the nebula with atomic weights 3 and 2 respectively. In the light of recent facts, if it be admitted that there are no elements with atomic weights between 1 and 4, the low values 3 and 2 obtained by interference methods have to be accounted for by supposing that

$$\phi({}_HA_0 l) > \phi({}_xA_0 l).$$

Since $\phi(A_0 l)$ diminishes as $A_0$ increases, $l$ remaining unaltered, the above inequality means that the absorption coefficients of the unknown elements for the lines used in the calculation are larger than the coefficient of luminous hydrogen in the nebula for $H_\gamma$.

Buisson and Fabry* by an ingenious interferometer method

* *Loc. cit.*; also Astrophysical Journal, xxxi. p. 98 (1910).

have measured the widths of some of the absorption lines in the solar spectrum.

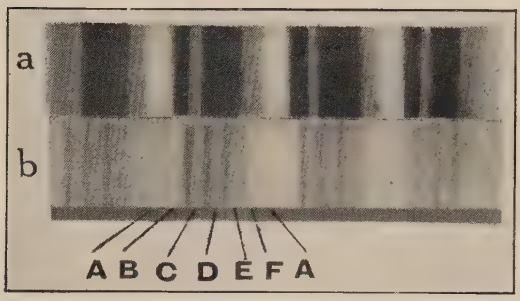
TABLE V.

| Radiation. | Temperature width at 6000°. | Observed width in solar radiation (Buisson & Fabry). |
|---|---|---|
| 6563 Å ($H_\alpha$) ..... | ·362 | ·960 |
| 4861 Å ($H_\beta$) ..... | ·270 | ·750 |
| 5896 Å ($D_1$) ..... | ·068 | ·160 |
| 5890 Å ($D_2$) ..... | | ·175 |
| Iron line at 4400 Å ... | ·033 | ·070 |

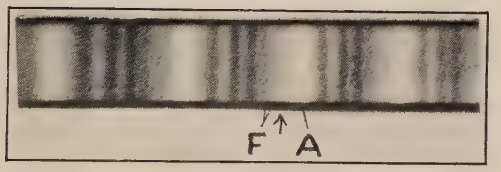
They account for the observed width by assigning part of the widening to the temperature (6000°) of the solar reversing layer, and part to the influence of the pressure there. Basing their calculations on the pressure-shift of the lines towards the red end of the spectrum, they arrived at 6 atmospheres as the value of the pressure in the reversing layer. In a subsequent paper *, while re-discussing their results, they attribute the shift of the solar lines to the Einstein gravitation effects, and consider the pressure in the reversing layer to be low. Fowler and Milne (Monthly Notices Roy. Astr. Soc. lxxxiii. p. 416, 1923), from considerations of the intensities of spectral lines in stellar spectra based on the theory of high-temperature ionization, conclude that the pressure in the reversing layer is of the order of $10^{-4}$ atmos. (about 0·1 mm. of mercury). In the paper of Fowler and Milne there is a summary of the available astronomical evidence in support of the view that the pressure in the reversing layer is low, some of these estimates being as low as $10^{-6}$ atmos., *i. e.* about ·001 mm. of Hg. If it be granted that the pressure in the reversing layer is small—say of the order of 0·1 mm. of mercury,—that part of the widening which was originally attributed to the effect of pressure remains unexplained. To take a case discussed by Buisson and Fabry, the observed width of a solar iron line at 4470 Å is ·070 Å, while the calculated value, taking into account temperature only, is ·033 Å at 6000°. If the effect of pressure in widening the solar lines is ruled out, the additional widening (viz. ·037 Å) has to be traced to the influence of the depth of the reversing layer. The increase in the observed width of other solar lines, such as $H_\alpha$ and $H_\beta$ of hydrogen and $D_1$ and $D_2$ of

* *C. R.* clxxii. p. 1020 (1921).

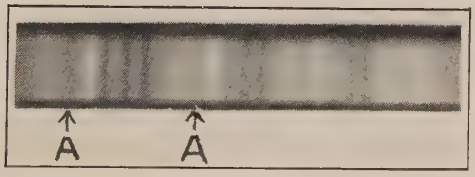
I.




II.




III.



sodium (see Table V.), over and above what can be ascribed to temperature, has to be accounted for in a similar manner. It is of interest to note in this connexion that an increase to 1000 miles of the length of the radiating column of luminous mercury vapour, under conditions of excitation considered in the calculations embodied in this paper, increases the width of a heavily-absorbed line of the 5461 Å group only to ·028 Å.

The mercury reasonance line 2537 Å is easily reversed in a prism spectroscope on the continuous background of the spectrum of white light which has passed through unexcited mercury vapour at 60° C. At this temperature the temperature width of the lines at 2537 Å is too small for the reversal to be revealed in such a low resolving power arrangement as a prism spectroscope ; but $A_0 l$, the product of the coefficient of absorption and the thickness of the absorbing layer (1 cm.) of normal mercury vapour at 60° C. (pressure ·02 mm.), is nearly 440, as may be calculated from the observation recorded by R. W. Wood that the intensity of the resonance radiation in mercury vapour at 20° C. (·001 mm. pressure) falls to 1/3 in a distance of 0·5 mm. This high value of $A_0 l$ has the effect of increasing the width of the absorption line; and if the line is complex and has very close satellites, the effective region of absorption is greatly increased as the absorption lines of such satellites widen out and fuse with the absorption line of the main component.

## *Description of Plate.*

Spectrogram II. was taken when the pressure in the small lamp (fig. 5, ii.) was 3 mm. The arrow-head shows the position of the separating line between A and F. This separating line is absent in the long-column pattern I. (*b*). Spectrogram III. was taken with the pressure kept low in the long column (fig. 5, i.) by the use of a liquid-air-cooled charcoal tube. The difference in the appearance of patterns II. and III. is distinct, the lines in the long column pattern III. appearing fuzzy and widened. The arrow-heads in III. show the positions in two consecutive orders of the unresolved main line, which has suffered comparatively greater absorption by the luminous vapour cooled by the neighbouring air-blast directed against the bulb $b_2$ (fig. 5). I. *b* and III. are long column patterns.

## V. *Absorption Spectrum of Potassium Vapour at High Temperatures.* *By* N. K. Sur, *M.Sc.*, *and* R. N. Ghosh, *M.Sc.*, *University of Allahabad* *.

[Plate II.]

SAHA'S well-known theory of thermal excitation of elements at high temperatures states that as the temperature of a mass of gas is gradually increased the gas is at first capable of absorbing the wave-lengths belonging to the primary series characteristic of the element. As the temperature is further raised the primary lines are emitted. At this stage there will be some atoms for which the valency electrons will be in the orbit of next higher energy level than that of normal position. Such atoms are called excited atoms. With the rise of temperature, the proportion of neutral atoms, which are only capable of absorbing the lines of the primary series, diminishes, while those in the excited condition of next higher energy level, viz. in the $2p$ orbit, increase. At this stage the characteristic wave-lengths of the subordinate series are capable of being absorbed by the heated gas. The intensity of the absorption lines belonging to the subordinate series can be easily calculated from Fowler and Milne's † extension of Saha's theory. Let the numerals 1, 2 denote the possible stationary states of a given neutral atom, arranged in order of decreasing negative energy; the numeral 1 denotes the atom in its normal state. Let $x_1$, $x_2$ denote the energy of the levels, so that $x_1$ is the ionization potential, $x_1 - x_2$ is the first resonance potential, etc. Further, let $q_1$, $q_2$ be the weights of corresponding states. Let $x$ be the fraction of atoms ionized; let $P_e$ be the partial pressure due to electrons from all sources, and let $\sigma$ be the number of valency electrons in the atoms in equivalent orbits, and let $f_r$ be the fraction of neutral atoms in the $r$th state. Then $f_r$ is given by the formula

$$f_r = q_r e^{-(x_1 - x_2)/kT} / b(T),$$

$$b(T) = q_1 + q_2 e^{-\frac{(x_1 - x_2)}{kT}} + q_3 e^{-\frac{(x_1 - x_3)}{kT}} .$$

The fraction $n_r$ of the total number of atoms which are

* Communicated by Prof. M. N. Saha.

† Monthly Notices of Royal Astro. Soc. vol. lxxxiii.

neutral and in the $r$th state is given by

$$n_r = f_r(1-x),$$

where $x$ is the fraction ionized.

$$n_r = f_r = q_r e^{-(x_1 - x_r)/kT} / b(T)$$

approximately. For monavalent atoms

$$q_1 = q_2 = \ldots\ldots = q_r = 1,$$

$$b(T) = 1,$$

$$f_r = e^{-\frac{h\nu}{kT}}.$$

| Temp. absolute. | $n_r$. |
|---|---|
| 700 | $4\times10^{-12}$ |
| 900 | $2{\cdot}5\times10^{-9}$ |
| 1100 | $4{\cdot}9\times10^{-8}$ |
| 1300 | $9{\cdot}5\times10^{-7}$ |

This theory has been put to test at least qualitatively by King * for a number of elements. He heated the substance under examination in a tube-resistance furnace of carbon to temperatures of about 3000° C. The absorption of the lines of the subordinate series by potassium vapour has been obtained by Narayan † by heating potassium in a steel tube raised to a high temperature by surrounding it with wire of suitable material and carrying electrical current. He also used Bunsen burners simultaneously to facilitate the increase of temperature. In view of the necessity for pure thermal excitation of the atoms, the experiment on potassium vapour has been repeated by heating it in a steel tube in a coal furnace described below.

### *Experiments on Potassium Vapour.*

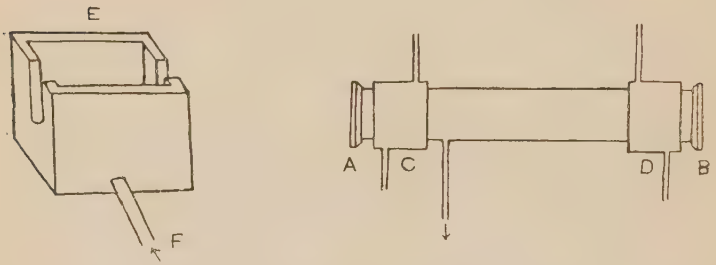
One of the earliest experiments on the absorption of potassium vapour is due to Bevan ‡. He recorded the absorption of the principal series in the ultra-violet down to the seventeenth number and observed the absorption bands

* Astrophysical Journal, lv. pp. 380–390 (1922).
† Narayan and Gunnayya, Phil. Mag. xlv. pp. 831–834.
‡ Proc. Phys. Soc. London, 1910; and Proc. Roy. Soc. lxxxiii. (1910).

in different parts of the spectrum, to which reference will be made later on. A more complete study of the principal series has been made by Datta *. The method adopted to obtain the reversal of the lines of the subordinate series is to vaporize the metal in an iron tube 40 cm. long and 3 cm. in diameter, as shown in figure 1.

The ends A and B were closed with plate glass sealed in brass sockets screwed to the ends of the tube. The joints were made air-tight by sealing-wax. To keep the ends cool, cold water was kept circulating through two outer jackets of brass C and D near the ends of the tube. The central portion of the tube containing the metal was heated strongly by surrounding it with a rectangular oven E made of clay and of suitable dimensions, through the end faces of which the tube could pass snugly, the top of the oven being kept

Fig. 1.




open for introducing coal. One of the side faces of the oven was fitted with a brass tube F, through which a strong blast of air could be admitted from a blower. By this means sufficiently high temperatures could be attained quickly. In fact, the temperature was high enough to melt the iron tube. To start with, the tube was always carefully exhausted by a Gaede oil-pump. If desired, the tube could be put in communication with the pump during the course of experiment. The source of light was a fifty candle-power point-o-lite lamp, and the spectrum was photographed on panchromatic plates in the first-order spectrum of a five-foot concave grating of dispersion 11·128 per mm. The wave-lengths were measured by means of a comparative spectrum of iron.

As soon as heating is begun dense vapour is formed, and the incident light is totally cut off. When the temperature of the tube is raised to 500° C. the colour of the transmitted

* Proc. Roy. Soc. ci. pp. 539–547 (1922).

light is green, and at the temperature of red heat the colour changes to bluish green. At this second stage no absorption line is observed in the visible region, but the spectrum is traversed by a number of channelled absorption bands. At the third stage, when the temperature is further increased, the absorption bands become more complicated and they are accompanied by one or two absorption lines scarcely visible on the plates. When the inner walls of the tube were raised to yellow heat a number of sharp absorption lines were obtained, and this state was found to be most suitable for obtaining reversals of lines of subordinate series. The temperature attained was about 1300° C., above which the iron tube melted. The following absorption lines were obtained :—

| Series notation. | *m*. | λ (Å.U.). |
|---|---|---|
| $2p—ms$ | 4 | 5802<br>5783 |
| ,, | 5 | 5340<br>5323 |
| ,, | 6 | 5099<br>5084 |
| $2p—md$ | 7 | 5112<br>5097 |

In addition to these, the following absorption lines were also obtained at high temperatures, but they could not be identified with known lines of potassium nor could they be allotted to any series relation. It may be noted here that some new lines have also been recorded by Datta *.

| λ (Å.U.). |
|---|
| 5225 |
| 5212 |
| 5206 |
| 5186 |
| 5167 |
| 5155 |
| 5136 |

The line $\lambda=4641$ $(1s—2d)$ was easily obtained on all the plates. That this is an exception to the principle of selection of Bohr has already been pointed out by Foote and Mohler † and by Datta ‡. The occurrence of the line in the present case where no external electric field is present supports their views.

* *Loc. cit.*
† Phil. Mag. xliii. (1922).
‡ *Loc. cit.*

*Bands.*

The presence of channelled bands in the absorption spectrum of potassium vapour has been noticed previously by Bevan * and Narayan †. Bevan noticed channelled bands in three distinct regions, (A) red, (B) orange-green, and (C) blue-violet. The edges of these absorption bands are regions of greater absorption and lie towards the violet. As in the case of the line spectrum the channelled bands also undergo variation with rise of temperature. First the bands of the red region appear in absorption, next, the bands in the blue-violet region, and lastly the bands in the orange-green region are obtained. These bands are clearly seen at low temperature (about 600° C.), but at higher temperature (about 1000° C.) they diminish in intensity. At intermediate temperatures the channelling tends to disappear and merges into wide bands. This is shown for the case of bands in the orange-green region in figs. 2 and 3 (Pl. II.). The D lines of sodium appear as impurity. Some measurements have been recorded both by Bevan and Narayan for the wave-lengths of the heads of the bands in the red region, but none of them has given any measurement for the bands in the other regions. A list of wave-lengths of the heads of the channelled bands in the orange-green region is recorded below.

| No. | λ (Å.U.). |
|---|---|
| 1 | 6024 |
| 2 | 6001 |
| 3 | 5979 |
| 4 | 5957 |
| 5 | 5923 |
| 6 | 5914 |
| 7 | 5870 |
| 8 | 5846 |
| 9 | 5824 |
| 10 | 5804 |
| 11 | 5784 |
| 12 | 5766 |
| 13 | 5746 |
| 14 | 5729 |
| 15 | 5714 |
| 16 | 5699 |
| 17 | 5682 |

Physics Department,
Muir Central College Block,
Univ. of Allahabad.

* *Loc. cit.* † *Loc. cit.*

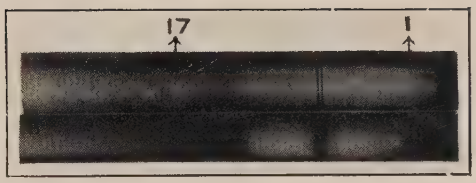



Fig. 2.

Fig. 3.

VI. *On the Depth of Earthquake Focus.* *By* SUDHANSU KUMAR BANERJI, *D.Sc., Director, Bombay and Alibag Observatories* *.

EVER since the late Dr. G. W. Walker † made the startling suggestion that the depth of earthquake focus is of the order one-fifth of the earth's radius, or about 1250 km., the subject has excited a great controversy ‡. It is not my intention in the present paper to enter into this controversy, or to discuss the accuracy of the values of the emergence angle of P waves measured at Pulkovo on which Walker relied for his conclusion. My object is to point out certain consequences of assuming a considerable depth for the focus on the different phases in the seismogram and to ask whether, in view of these consequences, it is reasonable to consider the focus to be very deep-seated. When the focus is very deep, any disturbance that starts there cannot at once initiate the surface waves. It can only give rise to three-dimensional longitudinal and transverse waves, which will travel to the observing station either by a single brachistochronic path or by several brachistochronic paths, after undergoing one or more reflexions at the earth's surface or in the heterogeneous mass of rocks constituting the earth's crust. The arrivals of all these waves are identified in the seismograms, and are designated as P, S, $PR_1$, $PR_2$, etc.; $SR_1$, $SR_2$, etc.; SP, PS, etc., waves. If the observing station is at a great distance from the focus it is not of much consequence, so far as the intensity of these waves is concerned, whether the depth of the focus is 100 km. or 1250 km., for the intensity will depend on the lengths of the brachistochronic paths. The maximum difference that is possible between the lengths of the brachistochronic paths which these waves will take to reach a very distant observing station, from two foci, one on the surface and the other at an internal point on the same diameter, is equal to the distance between the two foci. Consequently, even when the depth of the second focus is 1250 km., the amplitudes of

* Communicated by the Author.

† Phil. Trans. Roy. Soc. A, ccxxii. pp. 45–51 (1921).

‡ See, for instance, a note by the present writer in 'Nature,' Jan. 26, 1922; the Report of the discussion on the depth of earthquake foci held at the Meeting of the Royal Astronomical Society, March 3, 1922 ('The Observatory,' April 1922); and the Twenty-seventh Report of the Committee of Seismological Investigations, British Association, Hull, 1922.

the P or S waves received at a station very near the antipodes from the two foci will differ by a small quantity from each other.

This is, however, not the case with Rayleigh waves. Had the earth been an unlimited elastic solid, there would have been no surface reflexion and also no production of Rayleigh waves. But the earth has a spherical boundary, and consequently the portion of its surface known as the epicentral tract—that is to say, the region lying vertically over the internal focus—will be the first to be affected by the arrival of the longitudinal and transverse waves originated there. The disturbance in the epicentral tract will be maximum compared with the other parts of the earth's surface, the region being nearest to the focus, and will act as a secondary source for the production of surface waves. If it is then admitted that a part of the energy carried away by the longitudinal and transverse waves is transformed into surface waves on these waves reaching the surface, there is no reason to consider that the production of the surface waves would be confined to the epicentral tract alone, although no doubt the largest effects will be produced there. This is probably the reason why the long wave phase in the seismogram is a prolonged disturbance and is followed by a coda lasting for hours, particularly in the case of great tectonic earthquakes. The amplitudes of the longitudinal and transverse waves at any point in the epicentral tract will vary as inverse powers of the distance of the point from the focus. The depth of focus has therefore a considerable influence on the production of surface waves.

All records of distant earthquakes, if these have not been vitiated by the imposition of the free periods of the horizontal pendulum in the recording instrument, show that the amplitude of the long wave phase is generally much larger than the amplitudes of the other phases. This is what we should expect, because the surface waves being two-dimensional must acquire at a great distance from the source a continually increasing preponderance over the other waves. If, therefore, the effect of a considerable depth of focus on the production of Rayleigh waves is so great as to make the long wave phase in the seismograms at large epicentral distances merge into insignificance compared with P and S phases, then the hypothesis of a very deep focus will obviously have to be abandoned. I proceed, therefore, to investigate the effects of various focal depths on the relative importance of the different phases in the seismograms.

In the mathematical treatment of this problem, the source

of disturbance has to be assumed to be of a specified type, but may be assumed to be situated at any depth below the earth's surface. In an actual earthquake the source is not of the ideal type we represent by the usual mathematical forms. Even with simplified assumptions regarding the source, the mathematical difficulties in obtaining complete solutions of the elastic equations of motion consistent with the boundary condition, which is one of zero stress at the earth's surface, are very great. The displacements at any point on the earth's surface deduced from the solutions thus obtained agree, however, in their main features with the displacements recorded by seismographs. The effect of the complex nature of an earthquake source is mainly to introduce complexities in the character of the waves of the different phases in the seismograms and to prolong the disturbance in each phase, but not to alter the essential features by which these phases are recognized in the seismograms. No doubt, if the earthquake be supposed to originate at a fault, the mass movement there may be horizontal, inclined, or vertical, or of the nature of a twist; and this mass movement may influence one type of waves more than the others, and the energy sent out in different azimuths and at different vertical angles may be dependent on the direction of the fault and on the nature of the mass movement. But the law of divergence holds, and the two-dimensional waves always gain a continually increasing importance with distance over the three-dimensional waves, unless there are factors which make the immediate production of the two-dimensional waves impossible, and make them dependent for their initiation on the arrival of the three-dimensional waves at the surface.

### 2. *Seismic Waves due to a Source situated at the Earth's Surface.*

In order that any purely theoretical consideration can be made applicable to form an estimate of the depths of earthquake foci, it is essential first to prove that the kind of motion we obtain at any point on the earth's surface due to a single impulse of a specified type situated on the surface of the earth agrees in its main features with those observed in an earthquake. When this general similarity has been proved, we can then examine what happens when the focus is situated at a definite depth below the surface.

In a well-known paper published in the 'Philosophical Transactions of the Royal Society,' A, cciii. (1904), Professor Horace Lamb has investigated the propagation of

tremors over the surface of a semi-infinite elastic solid. He takes the source of disturbance to be a single arbitrary impulse of the type

$$R(t) = \frac{\overline{R}}{\pi} \frac{T}{t^2 + T^2},$$

situated at a point on the earth's surface, $t$ denoting the time, T the duration of impulse, and $\overline{R}$ a constant. This expression means that the pressure at the source changes from one constant value to another according to the law

$$A + B \tan^{-1} \frac{t}{T},$$

where A and B are constants. The disturbing pressure has no definite epoch of beginning or ending, but the range of time within which it is sensible can be made as small as we please by diminishing T.

The horizontal and vertical displacements of the ground due to a single impulse of this type have been calculated by Professor Lamb in pp. 37–40 of the paper above referred to, and the whole progress of these displacements has been plotted on p. 40. These show that as the wave system passes any point of the earth's surface, the horizontal displacement shows first of all a single well-marked oscillation on the arrival of the longitudinal waves (P). This is followed by a period of comparative quiescence, and then another oscillation corresponding to the epoch of arrival of transverse waves (S). The whole of this stage constitutes what is called the "minor tremor"; it is, of course, more and more protracted the greater the distance from the source, and its amplitude continually diminishes not only absolutely, but also relatively to that of the "main shock", which is identified with the arrival of the Rayleigh waves. Similar statements apply to the vertical displacements.

These results have been obtained for a semi-infinite elastic solid bounded by a plane, and cannot strictly apply to a spherical body like the earth. But then, if we confine our attention to the two principal waves P and S arriving at the station direct from the focus as well as the Rayleigh waves travelling over the earth's surface from the epicentral tract, and ignore for the present all waves which arrive at the station after undergoing a single or multiple reflexions and are recognized in the seismograms by the notation $PR_n$, $SR_n$, $S_nP_n$, $P_nS_m$, ($n=1, 2 \ldots$, $m=1, 2 \ldots$), the *chief qualitative*

*difference* introduced by the curvature of the earth would be that P and S waves would be (at distances comparable with the earth's radius) accelerated relatively to L. On account of the shorter routes along which the P and S waves will now have to travel, they will be a little more pronounced compared with L.

A comparison with the broad features of actual seismograms at large epicentral distances will make it clear that the figures given by Prof. Lamb on p. 40 of his paper give a reasonable representation of the effects due to a single impulse. The long wave phase appears to be a little more pronounced compared with the P and S phases than in actual seismograms at great distances, and this may partly be due to our neglecting the curvature of the earth and the source being supposed to be situated on the surface, and partly to the usual high preponderance of Rayleigh waves over the longitudinal and transverse waves when dissipation due to viscosity and other factors is neglected.

### 3. *Seismic Waves due to a Source situated at a Depth ($f$) below the Earth's Surface.*

The source of disturbance may now be supposed to be situated at a depth $f$ below the earth's surface. The origin of coordinates is supposed to be situated on the surface, the $z$-axis passing through the focus and the plane boundary coinciding with $z=0$. The form to be chosen for the source must be of a type which is possible in an elastic solid. Assuming first that the motion is simple harmonic ($e^{ipt}$), the elastic equations of motion are satisfied by

$$u=\frac{\partial\phi}{\partial x}+\frac{\partial^2\chi}{\partial z\,\partial x},\quad v=\frac{\partial\phi}{\partial y}+\frac{\partial^2\chi}{\partial y\,\partial z},\quad w=\frac{\partial\phi}{\partial z}+\frac{\partial^2\chi}{\partial z^2}+k^2\chi, \qquad . \quad . \quad . \quad (1)$$

where $u$, $v$, $w$ denote the components of velocity at any point, and $\phi$ and $\chi$ are given by

$$(\nabla^2+h^2)\phi=0,\qquad (\nabla^2+k^2)\chi=0, \qquad . \quad . \quad (2)$$

where

$$h^2=\frac{p^2\rho}{\lambda+2\mu}=p^2a^2,\qquad k^2=\frac{p^2\rho}{\mu}=p^2b^2, \quad . \quad . \quad (3)$$

$\rho$ denoting the density, and $\lambda$, $\mu$ the usual elastic constants, $a$, $b$ the *reciprocals* of the wave velocities corresponding to the irrotational and equivoluminal types of disturbance respectively. The source of disturbance at the point

$(0, 0, f)$, distance $f$ below the earth's surface, may therefore be assumed to be of the type

$$\phi = \frac{e^{-ihr}}{r}, \quad \chi = 0, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$r$ denoting the distance from this point.

The values of $\phi$ and $\chi$ at any point have then to be obtained consistent with a source of this type and the condition of zero stress on the plane boundary. This has been done by Prof. Lamb. The horizontal and vertical displacements at any point on the earth's surface are calculated by (1), and these are given by

$$q_0 = 4\int_0^\infty \frac{k^2\xi^2\beta}{\mathrm{F}(\xi)} e^{-\alpha f} \mathrm{J}_1(\xi\varpi)\, d\xi, \quad . \quad . \quad . \quad (5)$$

$$w_0 = -2\int_0^\infty \frac{k^2\xi(2\xi^2-k^2)}{\mathrm{F}(\xi)} e^{-\alpha f} \mathrm{J}_0(\xi\varpi)\, d\xi,$$

where

$q_0$ = horizontal displacement, $w_0$ = vertical displacement,

$\alpha^2 = \xi^2 - h^2$, $\beta^2 = \xi^2 - k^2$, $\varpi = \sqrt{(x^2+y^2)}$, and

$$\mathrm{F}(\xi) = (2\xi^2-k^2)^2 - 4\xi^2\alpha\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For a detailed analysis of this case, reference is made to Prof. Lamb's paper (pp. 33–36).

These are the complete expressions for the horizontal and vertical displacements at any point on the earth's surface, and comprise within themselves all the different phases of seismograms. In order, therefore, to fully interpret these integrals, these have to be broken up into terms of different orders of magnitude.

The generalization of these results so as to apply to an arbitrary time variation of the source, as has been pointed out by Lamb, follows much the same course as in article 9 of his paper. But so far as the effect of depth on the relative amplitudes of the three principal phases is concerned, the type of the source is probably not of much consequence.

In order to understand fully what the expressions (5) mean, we may enter into a little more detail.

If we assume the existence of an equal source at the image point $(0, 0, -f)$, and denote distance from this point by $r'$, and take

$$\phi = \frac{e^{-ihr}}{r} + \frac{e^{-ihr'}}{r'}, \quad \chi = 0, \quad . \quad . \quad . \quad . \quad (7)$$

then these expressions will account for all the reflexions of

longitudinal and transverse waves that occur on account of the existence of a bounding surface. But these do not satisfy the condition of zero stress on the boundary. Now, in the neighbourhood of the plane $z=0$,

$$\phi = \int_0^\infty \frac{e^{-\alpha(z+f)}}{\alpha} J_0(\xi\varpi)\xi\, d\xi + \int_0^\infty \frac{e^{\alpha(z-f)}}{\alpha} J_0(\xi\varpi)\xi\, d\xi$$

$$= \int_0^\infty \frac{\cosh \alpha z}{\alpha} e^{-\alpha f} J_0(\xi\varpi)\xi\, d\xi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Consequently, if $[p_{zz}]_0$ and $[p_{z\varpi}]_0$ denote the vertical component of the stress and the component of the stress in any horizontal direction respectively on this plane, we have

$$[p_{zz}]_0 = 2\mu \int_0^\infty \frac{(2\xi^2 - k^2)}{\alpha} e^{-\alpha f} J_0(\xi\varpi)\xi\, d\xi, \quad . \quad . \quad (9)$$

$$[p_{z\varpi}]_0 = 0.$$

In order, therefore, to annul the stresses on the plane $z=0$, we have to assume the existence of a system of negative vertical stresses of the type

$$-2\mu \frac{2\xi^2 - k^2}{\alpha} e^{-\alpha f} J_0(\xi\varpi)\xi\, d\xi \quad . \quad . \quad . \quad (10)$$

distributed over the plane, find out the displacements corresponding to these latter stresses, and then integrate with respect to $\xi$ from 0 to $\infty$. The displacements * at any point on the surface due to this distribution of stresses are

$$q_0 = \int_0^\infty \frac{\xi(2\xi^2 - k^2 - 2\alpha\beta)}{F(\xi)} \cdot \frac{2\xi^2 - k^2}{\alpha} e^{-\alpha f} J_1(\xi\varpi)\xi\, d\xi,$$

$$w_0 = -2\int_0^\infty \frac{k^2(2\xi^2 - k^2)}{F(\xi)} e^{-\alpha f} J_0(\xi\varpi)\xi\, d\xi. \quad . \quad . \quad . \quad (11)$$

These, when added to the displacements derived from (7), give the complete expressions already given in (5) for the displacements at any point on the earth's surface consistent with the type of source assumed and the condition of no stress on the boundary.

We now understand the genesis of the Rayleigh waves. Mere reflexions at the bounding surface of the waves generated at the internal source do not explain the whole phenomenon. Something more is required to satisfy the conditions which must hold at a free surface. The satisfaction

* See p. 32 of Prof. Lamb's paper.

of these conditions requires the imposition of a properly distributed system of stresses at the boundary surface, the imposed stresses accounting for the production of surface waves.

### 5. *Effect of the Depth of Focus on the Rayleigh Waves.*

We now proceed to an analysis of the complete expressions (5) for the displacements at any point on the surface due to an *internal* source situated at a depth $f$ below the surface, and to an examination of the effect of depth on the amplitude of Rayleigh waves.

The parts * of $q_0$ and $w_0$ which are most important are

$$\left.\begin{aligned} q_0 &= -\frac{4\pi k^2\kappa\beta_1}{\mathrm{F}'(\kappa)}\, e^{-\alpha_1 f}\,\mathrm{D}_1(\kappa\varpi)\, e^{ipt} + \text{ etc.},\\ w_0 &= \frac{2\pi k^2\kappa}{\mathrm{F}'(\kappa)}\,(2\kappa^2-k^2)\, e^{-\alpha_1 f}\,\mathrm{D}_0(\kappa\varpi)\, e^{ipt} + \text{etc.}, \end{aligned}\right\} \quad . \quad (12)$$

where $\kappa$ is a root of the equation $\mathrm{F}(\xi)=0$; that is,

$$1-8\frac{\xi^2}{k^2}+\left(24-16\frac{h^2}{k^2}\right)\frac{\xi^4}{k^4}-16\left(1-\frac{h^2}{k^2}\right)\frac{\xi^6}{k^6}=0, \quad . \quad (13)$$

and $\alpha_1$, $\beta_1$ are given by

$$\alpha_1{}^2=\kappa^2-h^2, \qquad \beta_1{}^2=\kappa^2-k^2.$$

At a great distance $\varpi$, we have

$$\left.\begin{aligned} q_0 &= \frac{4\pi i k^2\kappa\beta_1}{\mathrm{F}'(\kappa)}\, e^{-\alpha_1 f}\sqrt{\frac{2}{\pi k\varpi}}\, e^{i(pt-\kappa\varpi-\frac{1}{4}\pi)},\\ w_0 &= \frac{2\pi k^2\kappa(2\kappa^2-k^2)}{\mathrm{F}'(\kappa)}\, e^{-\alpha_1 f}\sqrt{\frac{2}{\pi k\varpi}}\, e^{i(pt-\kappa\varpi-\frac{1}{4}\pi)}. \end{aligned}\right\} \quad . \quad (14)$$

We easily identify that these expressions represent the displacements due to Rayleigh waves. It is important to notice the occurrence of the factor $e^{-\alpha_1 f}$ in these expressions. This represents the effect of the depth of the focus. The amplitude is seen to diminish with increasing distance according to the usual law ($\varpi^{-\frac{1}{2}}$) of annular divergence.

The influence of the factor $e^{-\alpha_1 f}$ on the amplitude of the Rayleigh waves may now be examined in detail. If the material of the earth be supposed to satisfy Poisson's condition between the elastic constants, namely $\lambda=\mu$, then $h^2=\frac{1}{3}k^2$. The cubic equation (13) then has all its roots real.

* See p. 36 of Lamb's paper, expressions (161).

It has been shown by Rayleigh that only one of these three roots gives rise to real Rayleigh waves. This root is

$$\frac{\kappa}{k} = 1{\cdot}087664\ldots.$$

We have then

$$\alpha_1^2 = \kappa^2 - h^2 = \left[\left(\frac{\kappa}{k}\right)^2 - \frac{1}{3}\right] k^2$$

$$= \left[(1{\cdot}088)^2 - \tfrac{1}{3}\right] k^2 = (0{\cdot}92k)^2.$$

Therefore

$$\alpha_1 = 0{\cdot}92k = 0{\cdot}92 \times \frac{2\pi b}{\mathrm{T}},$$

where T = the period of Rayleigh waves, mainly 12 seconds and 20 seconds, and

$\frac{1}{b}$ = the velocity of propagation of transverse waves

= 4 km. per second.

Taking T = 12 seconds, we find $\alpha_1 = 0{\cdot}12$,

and T = 20 seconds, we find $\alpha_1 = 0{\cdot}08$.

We therefore take $\alpha_1$ to be roughly equal to $\frac{1}{10}$.

If, now, A denotes the amplitude of the Rayleigh waves recorded at any station, produced by an earthquake the focus of which is situated on the surface, then, when the focus is

(1) 10 km. deep, the amplitude = $Ae^{-1}$,

(2) 100 km. deep, the amplitude = $Ae^{-10}$,

(3) 1000 km. deep, the amplitude = $Ae^{-100}$,

the sources being supposed to be of the same intensity and all in the same vertical line. Since $e = 2{\cdot}72$ nearly, the amplitude of the Rayleigh waves undergoes a very rapid diminution indeed with increasing depth of the focus.

### 5. *Effect of the Depth of Focus on the* P *and* S *Waves.*

It has already been pointed out that the most important terms in the complete expressions (5) for the displacements of any point on the surface are those numbered (12). The general form and the order of magnitude of the residual disturbance have also been obtained by Lamb* when $h\varpi$ and $k\varpi$ are large, $\varpi$ denoting as usual the distance from the

* See p. 36 of his paper.

axis. The lowest order terms depending on $\frac{1}{k\varpi}$ in the residual expressions for $q_0$ and $w_0$ are of the types

$$\frac{e^{ipt}}{(k\varpi)^{\frac{3}{2}}}\int_0^\infty \frac{e^{-ik\varpi\cosh u}}{(\cosh u)^{\frac{1}{2}}}\,du \quad \text{and} \quad \frac{e^{ipt}}{(k\varpi)^{\frac{3}{2}}}\int_0^\infty \frac{e^{-ik\varpi\cosh u}}{(\cosh u)^{\frac{3}{2}}}\,du.$$

By considering the asymptotic expansion of these integrals, Prof. Lamb has shown these to be ultimately comparable with

$$e^{ip(t-b\varpi)}/(k\varpi)^2,$$

which therefore represents the most important term in the transverse waves (S). In the same way the terms in $q_0$ and $w_0$ which correspond to the longitudinal waves are ultimately comparable with

$$e^{ip(t-a\varpi)}/(k\varpi)^2.$$

The depth of the focus does not enter into these expressions, but the amplitude diminishes inversely as the square of the distance from the source. This is simply because there is a boundary. Had the elastic waves simply diverged from a centre in an unlimited medium, the amplitude would have varied inversely as the distance.

It is not difficult to understand why the effect of the depth of focus on the P and S phases at large epicentral distances (large compared with the focal depth) is negligible. Whenever there is a boundary, the law of divergence for the P and S waves is always the inverse square of the distance. Since the maximum difference possible between the distances of an observing station from a focus situated on the earth's surface, and one at a depth $f$ below, is the distance between the foci, the maximum value of the ratio of the amplitudes of the P or S waves arriving at the station from the two foci (supposed to be of equal strengths) is

$$\frac{\varpi^2}{(\varpi - f)^2} = 1 + \frac{2f}{\varpi} = 1, \quad \text{approximately,}$$

in all cases where $\frac{f}{\varpi}$ can be regarded as small.

### 6. *Possible Depths of Earthquake Foci.*

In the seismograms we have a ready means of determining the relative amplitudes of the different phases. The records of all aperiodic seismographs for distant earthquakes definitely show that the amplitudes of the L phase are much

greater than those of the P and S phases. If, therefore, we can prove that when the focus is 1000 km. deep (say) the seismogram at a distant observing station ought to show the amplitude of the L phase to be much smaller than those of the P and S phases, then, this being contrary to the facts, we must conclude that 1000 km. is too large a depth for the focus.

Let us take an extreme case, but a typical one. Suppose that there are two sources of disturbance of equal strengths, one at the earth's surface and the other at a depth of 1000 km. Let the observing station be at a distance of 20,000 km. measured circumferentially from the first focus—that is to say, very near the antipodes, the earth's radius being 6500 km. (say). If we are permitted to regard the sources as simply point-sources, and if we suppose that the amplitudes of P, S, and L waves produced by the surface source at unit distance (here 1 km.) from it are all the same and equal to A (say), then the amplitudes of these waves at the observing station will be

$$\frac{A}{(13{,}000)^2}, \quad \frac{A}{(13{,}000)^2}, \quad \frac{A}{(20{,}000)^{\frac{1}{2}}}$$

respectively, P and S waves being supposed to have travelled by straight paths passing almost through the earth's centre.

This means that although at a distance of 1 km. the amplitudes of all these waves were the same, when these waves have reached the observing station, the amplitudes of the L waves are about $12 \times 10^5$ times greater than the amplitude of the P or S waves.

The amplitudes of the waves received from the second focus will, on the other hand, be

$$\frac{A}{(13{,}000-1000)^2}, \quad \frac{A}{(13{,}000-1000)^2}, \quad \frac{Ae^{-100}}{(20{,}000)^{\frac{1}{2}}}$$

respectively. That is to say, the conditions are now reversed, and the amplitudes of the P or S waves are $2{\cdot}688 \times 10^{37}$ times greater than the amplitudes of the Rayleigh waves. It is inconceivable that there is any factor we have ignored which can restore this enormous reduction of the amplitudes of the Rayleigh waves and make them larger than the amplitudes of the P or S waves. A depth of 1000 km. is therefore impossible for an earthquake focus.

If the depth of the second focus is only 100 km., the

amplitudes of the P, S, and L waves at the observing station will be

$$\frac{A}{(13{,}000-100)^2}, \qquad \frac{A}{(13{,}000-100)^2}, \qquad \frac{Ae^{-10}}{(20{,}000)^{\frac{1}{2}}}$$

respectively.

This makes the amplitude of the Rayleigh waves about 55 times greater than the amplitude of the P or S waves. 100 km. is therefore a possible depth for an earthquake focus.

A focal depth of 200 km. will make the amplitude of the Rayleigh waves received at the observing station about $\frac{1}{400}$ of the amplitude of either P or S waves. A depth of even 200 km. will therefore appear to be too large an estimate for the earthquake focus.

### 7. *Discussion of the Problem when the Earth is assumed to be Spherical.*

In a recent paper Jeans* has discussed the problem, particularly in certain of its aspects which seem to have a direct seismological importance, of the vibrations of an elastic sphere in which the mutual gravitational attraction of the parts is taken into account as well as the variation of the physical constants of the earth's material with distance from the centre. The free vibrations are deducible from the functions of the type

$$\left.\begin{aligned} P &= F_1(r)S_n(\theta,\phi)\cos pt, \\ w' &= f_1(r)S_n(\theta,\phi)\cos pt, \end{aligned}\right\} \quad \text{(Primary vibrations)},$$

$$S = F_2(r)S_n(\theta,\phi)\cos pt, \quad \text{(Secondary vibrations)},$$

where $S_n(\theta,\phi)$ is a surface harmonic of degree $n$, and $F_1(r)$, $f_1(r)$, $F_2(r)$ satisfy certain differential equations with respect to $r$. A natural earthquake shock can be regarded as being made up in great part of free vibrations in which the order $n$ of the surface harmonic is large, which is tantamount to neglecting all earthquake waves except those of wave-length short in comparison with the earth's radius. With this assumption the solution for S, throughout any small range of values of $r$ in the neighbourhood of $r=a-f$, can be written in the form

$$S = B\exp.\left[-r\left\{n(n+1)\left(\frac{1}{(a-f)^2}-\frac{1}{a^2}\right)\right\}^{\frac{1}{2}}\right] \times\cos\{[n(n+1)]^{\frac{1}{2}}\theta \pm pt\},$$

* Jeans, Proc. Roy. Soc. A, cii. pp. 554–574 (1923).

where

$$p = \frac{1}{a}\sqrt{\left[n(n+1)\frac{\mu}{\rho}\right]}, \quad a = \text{earth's radius}, \quad f = \text{focal depth},$$

the focus being supposed to be situated on the radius $\theta = 0$. This can also be written in the form

$$S = B \exp.\left[-\frac{r}{a}\left\{2n(n+1)\frac{f}{a}\right\}^{\frac{1}{2}}\right] \cos\left\{\left[n(n+1)\right]^{\frac{1}{2}}\theta \pm pt\right\},$$

showing that the amplitude at any point on the earth's surface, $r = a$, decreases with respect to the focal depth according to the exponential law

$$e^{-\alpha(f/a)^{\frac{1}{2}}},$$

where

$$\alpha = \sqrt{2n(n+1)}.$$

As shown by Jeans, these surface waves represent a complete system of free vibrations, and so travel round and round the globe without any loss of energy except that due to dissipative forces such as viscosity; there is no tendency for these waves to diffuse into the earth's interior. Now,

$$\alpha = \sqrt{2} \cdot \frac{ap}{v},$$

$v$ being the velocity of propagation of the waves considered; that is to say,

$$\alpha = \frac{2\pi a}{Tv} \cdot \sqrt{2} = \frac{40{,}000}{20 \times 4}\sqrt{2} = 707 \text{ nearly},$$

taking $T = 20$ seconds, $v = 4$ km. per second, and $2\pi a =$ circumference of the earth $= 40{,}000$ km. If, then, A denotes the amplitude of the surface waves recorded at any station produced by an earthquake the focus of which is situated on the surface, then, when the focus is

(1) 10 km. deep, the amplitude $= Ae^{-28}$,

(2) 100 km. deep, the amplitude $= Ae^{-88}$,

(3) 1000 km. deep, the amplitude $= Ae^{-277}$.

This shows a more rapid diminution of amplitude of surface waves with respect to the focal depth than what was found in art. 4. When these results are used with the arguments employed in art. 6, it would appear that a depth of even 100 km. is too much for an earthquake focus.

To obtain the surface waves caused by the primary

disturbance, we assume that in the neighbourhood of $r=a-f$ the solutions of the differential equations (60) and (62) in Jeans's paper are

$$w' = Ae^{-lr}\cos\{[n(n+1)]^{\frac{1}{2}}\theta \pm pt\},$$
$$P_0 = Be^{-mr}\cos\{[n(n+1)]^{\frac{1}{2}}\theta \pm pt\},$$

where

$$l^2 = \frac{n(n+1)}{(a-f)^2} - p^2\frac{\rho}{\mu}, \qquad m^2 = \frac{n(n+1)}{(a-f)^2} - p^2\frac{\rho}{\lambda+2\mu},$$

so that the boundary equation (65) in Jeans's paper becomes

$$lm\frac{n(n+1)}{a^2} = \left[\frac{n(n+1)}{a^2} - \tfrac{1}{2}p^2\frac{\rho}{\mu}\right]^2.$$

Squaring both sides of this equation and substituting for $l^2$ and $m^2$, we obtain

$$\left[\frac{n(n+1)}{(a-f)^2} - p^2\frac{\rho}{\mu}\right]\left[\frac{n(n+1)}{(a-f)^2} - p^2\frac{\rho}{\lambda+2\mu}\right]\frac{n^2(n+1)^2}{a^4} = \left[\frac{n(n+1)}{a^2} - \tfrac{1}{2}p^2\frac{\rho}{\mu}\right]^4.$$

When $f=0$, this equation simplifies to

$$(\lambda+2\mu)\theta^3 - 8(\lambda+2\mu)\theta^2 + 8(3\lambda+4\mu)\theta - 16(\lambda+\mu) = 0,$$

where

$$p^2 = \frac{n(n+1)}{a^2}\cdot\frac{\mu}{\rho}\theta.$$

These surface waves then become identical with the usual Rayleigh waves. Now,

$$l^2 = \frac{n(n+1)}{a^2}\left(1+\frac{2f}{a}\right) - p^2\frac{\rho}{\mu}$$
$$= \frac{n(n+1)}{a^2}\left[1+\frac{2f}{a} - p^2\frac{\rho}{\mu}\frac{a^2}{n(n+1)}\right]$$
$$= \frac{n(n+1)}{a^2}\left[1-\theta+\frac{2f}{a}\right],$$

and

$$m^2 = \frac{n(n+1)}{a^2}\left(1+\frac{2f}{a}\right) - p^2\frac{\rho}{\lambda+2\mu}$$
$$= \frac{n(n+1)}{a^2}\left[1-\frac{8}{17}\theta+\frac{2f}{a}\right],$$

since for the earth $\lambda/\mu$ is probably about $1\frac{1}{8}$.

For these types of waves, then, the laws of diminution of amplitude with respect to the focal depth are

$$e^{-\alpha\left(\frac{1-\theta}{2}+\frac{f}{a}\right)^{\frac{1}{2}}} \text{ and } e^{-\alpha\left(\frac{17-8\theta}{34}+\frac{f}{a}\right)^{\frac{1}{2}}}$$

for $\omega'$ and $P_0$ respectively,
where $\alpha$ has the same value as before, namely

$$\alpha = \sqrt{2n(n+1)}.$$

Now $\sqrt{\theta}=0{\cdot}9218$ and we get therefore for these waves a slightly more rapid diminution of amplitude with respect to the focal depth than what we found in art 4.

### 8. *Experimental Evidence in support of "High" and "Deep" Focus.*

A method for estimating the depth of focus, based on an empirical formula for the reception of primary waves at stations near the anticentre, has been recently developed by Prof. Turner *. The method suggests a normal depth $d$ for the focus, which has some such value as 0·04 of the earth's radius. Several earthquakes have been studied by Turner by this method, and focal depths both above and below the normal depth have been assigned to them. There is some uncertainty about the value of $d$, but the following cases of "high" focus will not be inconsistent with the results obtained in this paper :—

| Year. | Date. | | | | Epicentre. | Focal Depth. |
|---|---|---|---|---|---|---|
| | d | h | m | s | | |
| 1918 ..... Sept. | 7 | 17 | 15 | 51 | 46°·5 N., 151°·4 E. | $d-{\cdot}030$ |
| ,, ..... ,, | 8 | 0 | 9 | 30 | ,, ,, | $d-{\cdot}030$ |
| ,, ..... ,, | 11 | 4 | 6 | 5 | ,, ,, | $d-{\cdot}030$ |
| 1919 ... ..... May | 6 | 19 | 40 | 45 | 6°·0 S., 153°·0 E. | $d-{\cdot}030$ |

The values obtained for the depths of the "deep" focus are inappropriate with the conclusions of this paper.

Dr. Dorothy Wrinch, by an altogether different procedure, finds that the observations of the Calabrian earthquake of 1905, September 8, taken at stations at distances of between 40 and 220 km., give 40 to 50 km. as an upper estimate of the depth of its focus, in agreement with the results obtained in this paper. It will perhaps be better to postpone criticism of the experimental results so far obtained till a method has been determined for finding out the actual depth instead of the relative depth of the focus from the observed data ; but the suggestion may be put forward that the systematic deviation

* 'Monthly Notices of the Royal Astronomical Society,' Geographical Supplement, vol. i. no. 1 (1922).

from the time of arrival of the first waves as given by the empirical formula of Turner, which any particular earthquake is liable to show at all antipodal stations, may to a certain extent be due to the effect of the rotation of the earth which the formula does not take account of. The effect will be negative or positive, according as the wave systems travel in or against the direction of rotation of the earth. It is proposed to discuss this effect in a subsequent paper.

### 9. *Summary and Conclusion.*

The conclusions of this paper may be summarized as follows. When the earth is assumed to be an infinite solid bounded by a plane, it is found that—

(1) The amplitudes of the long wave phase in the seismogram undergo a very rapid diminution with increasing depth of focus. If $f$ = the depth of focus, the amplitude is proportional to $e^{-\alpha_1 f}$, and when $f$ is expressed in kilometres, the value of $\alpha_1$, under typical conditions, is $\frac{1}{10}$.

(2) The effect of the depth of the focus on the amplitudes of the P and S waves at large epicentral distances is almost negligible.

(3) If the depth of the focus is 1000 km., then the seismogram at a station very near the anticentre should show the amplitude of the P and S waves about $2{\cdot}7 \times 10^{37}$ times larger than the amplitude of the L waves, assuming that all these waves had the same amplitude at unit distance from the source. This is contrary to facts, and therefore a depth of 1000 km. is too large an estimate for the depth of earthquake focus.

(4) If the depth is 200 km., then the amplitude of P and S phases at the same station would be 400 times larger than the amplitude of the L phase. The earthquake focus is therefore probably not even 200 km. deep.

(5) If the depth is 100 km., then the amplitude of the L waves is about 55 times larger than the amplitude of the P or S waves. Roughly speaking, therefore, 100 km. or anything in the neighbourhood of 100 km. or less than 100 km. is a possible depth for the earthquake focus.

When the earth is assumed to be a perfectly spherical body, it is found that the effect of the focal depth on the amplitude of the Rayleigh waves becomes more accentuated, and the results indicate that the depth is probably less than 100 km.

My best thanks are due to Sir Gilbert T. Walker, F.R.S., for the interest he has taken in the paper.

The Observatory,
Colaba, Bombay,
July 7, 1924.

# VII. *Turbulence and Vertical Temperature Difference near Trees.* By Lewis F. Richardson, *B.A., F.Inst.P.**

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§ 6. Personal.
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## § 1. *Introduction.*

WHEREAS Osborne Reynolds's criterion of turbulence agrees fairly well with the behaviour of the fluid in a pipe, yet in the open air it is known that the up-gradient of temperature exerts in addition a strong controlling influence. This problem is explored by observations, which are brought into relation with a theory already published.

## § 2. *The Vertical Difference of Temperature.*

In order to measure the temperature-difference, a thermo-electric arrangement was set up (as shown in fig. 1) partly on an iron mast, partly on a tree. The junctions $\alpha$ and $\beta$ were 4·4 and 18·3 metres above the ground. Radiation and rain were screened off by horizontal zinc plates, two above and one below each junction. The uppermost plate was about 30 cm. in diameter. This type of screen was a modification of one devised by the Indian Survey Department. A thick eureka wire passed directly from one screen to the other, and from its ends copper wires were led to a very sensitive moving coil galvanometer of 1·9 ohms resistance by Hartmann & Braun.

The arrangement was standardized by preparing two vacuum jacketed flasks containing paraffin oil at slightly different temperatures. One junction was placed in each flask, the galvanometer was read, the flasks interchanged, and the reading repeated. The temperature difference of the oil was taken before and after by mercury thermometers. Five comparisons were also made with an aspirated thermometer carried up and down the mast. These readings confirmed in a rough way the standardization by oil, but, as they were not simultaneous at top and bottom, they are

* Communicated by the Author. Read at the British Association for the Advancement of Science, Toronto, 1924.

considered to be less accurate. The galvanometer could be read to 0°·003 C., and was certainly reliable to 0°·03 C.

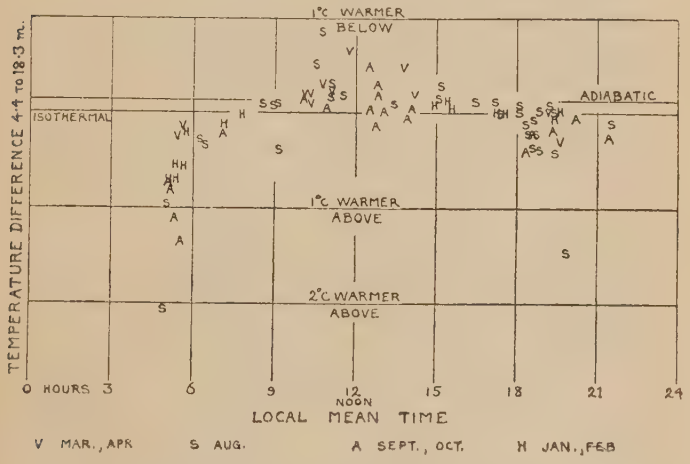
Care was taken not to record any observation when either junction was likely to be wet by rain, nor when horizontal sunbeams fell on it.

Fig. 1.

Fig. 2 shows the diurnal variation of the difference of temperature. If the zinc radiation-screens were inadequate, then the upper junction would on the average be warmest, relatively to the lower junction, at mid-day when sun could shine directly on the zinc plates of the upper junction, while the lower junction was under the shade of a fir-tree. Actually the reverse is observed, hyper-adiabatic gradients being frequent between 10 h. and 16 h. G.M.T. These

facts lead me to believe: firstly, that the screening was efficient; and secondly, that hyper-adiabatic gradients did really exist. At the times when such gradients were recorded the galvanometer oscillated irregularly, as we should expect if, owing to the instability, the warm air was rising in streaks and patches.

Fig. 2.




## § 3. *The Mean Wind and the Gusts.*

The wind velocity and direction were measured by a Dines anemometer at a height of 26 m. above the ground, and at a height of 2 m. by a small windmill anemometer also of Mr. W. H. Dines' design.

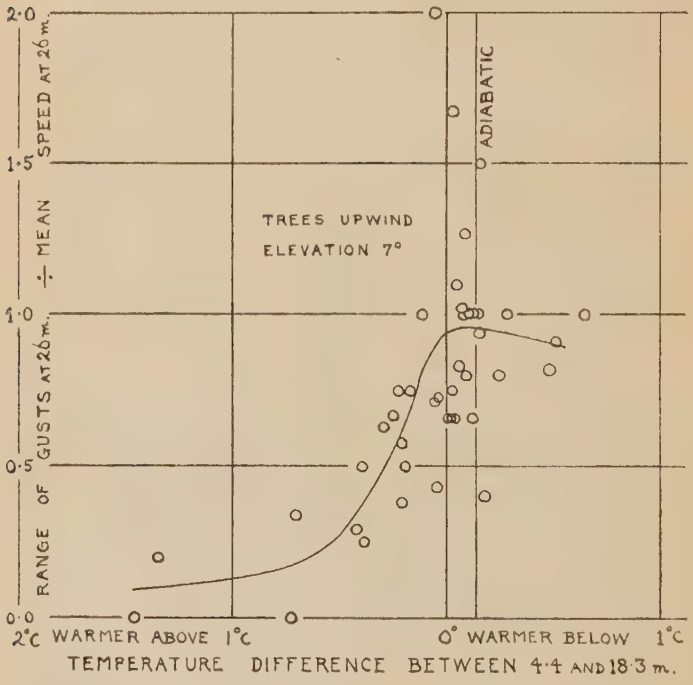
The turbulence at 26 m. was indicated by the range of gusts shown on the trace of the P.T. anemometer. By gusts are meant those variations of wind which cause the anemometer pen to go to and fro many times within an interval of ten minutes. The slower variations having periods lying between say 10 minutes and an hour are not discussed here. It being common knowledge among meteorologists that the range of gusts is, on the average, proportional to the mean velocity at the same level, the ratio of the former to the latter has been given the name "gustiness," and has been a subject of various studies.

Fig. 3 exhibits the gustiness at 26 m. as a function of the difference of temperature in a layer below that level. In spite of the considerable scatter, which may be due to the different exposures in directions here lumped together, it is evident that the gustiness varies on the average somewhat

in the way indicated by the line drawn on the diagram. As we should expect, the gustiness is small when the air is much colder below, and increases as the labile adiabatic equilibrium is approached. It is surprising, however, to observe that, as far as the scanty observations go, the unstable hyperadiabatic condition did not correspond to still further increased gustiness.

An analogous peculiarity has been noted in the upper air, as may be seen from the dotted line in the diagram on p. 78 of "Weather Prediction by Numerical Process" *.

Fig. 3.




Let us now consider winds from directions in which the trees were much further away, subtending an elevation of only 1° at the instruments instead of 7°. It was found that for adiabatic conditions:—

| | | |
|---|---|---|
| Mean wind at 26 m. ..... | 7·5 m/s. | 4 m/s. |
| Range of gusts at 26 m. ..... | 5 m/s. | 2·5 m/s. |
| Gustiness $= \frac{\text{range of gusts}}{\text{mean wind}}$ ... | 0·67 | 0·62 |

* Cambridge University Press.

It is seen that on fig. 3 the corresponding ratio is 0·95 when the trees subtend 7°.

§ 4. *The Theoretical Criterion of Just-no-turbulence.*

Another way of regarding the turbulence, a way which is less simple but more likely to give us insight into the processes at work, will be to bring the observations into relation with a certain "criterion" deduced from theory, in a paper* on the Supply of Energy from and to Atmospheric Eddies. It may help to clear ideas if I repeat in a simple form the fundamental notions of the previous paper, especially as it dealt with the whole thickness of the atmosphere, whereas our present observations are in a layer comparatively thin.

Imagine a horizontal slab of air bounded by fixed pressures $p+\delta p$ and $p$. Its mass per area is $\delta p/g$, and remains fixed as it is warmed. If $\theta$ denotes its absolute temperature and $\gamma_v$ denotes the specific heat of air at constant volume, then the rate at which the slab acquires thermal energy is denoted by $(\delta p/g)\gamma_v . \partial\theta/\partial t$, where $t$ is the time. We shall suppose that specific heats are expressed with ergs not calories. As the slab is warmed, it swells, lifting the air above, which therefore acquires gravitational energy. It may be shown that the rate of increase of this gravitational energy is $(\delta p/g) . b\partial\theta/\partial t$, where $b$ is the gas-constant in $p=b\rho\theta$. This is also evident because $\gamma_v+b=\gamma_p$, the specific heat at constant pressure, and because we can include both effects by simply thinking of the air as being warmed at constant pressure. If the material of the slab were a solid such as copper or marble, then, as Fourier stated, the excess of the heat flowing in at one face over the heat flowing out at the other just accounts for the rate of increase of the heat contained in the slab. No heat appears or disappears. But if the slab is air and the flux of heat is due to eddy-conductivity, then the excess of the flux into, over the flux out of, the slab does not in general equal the rate of gain of thermal energy in the slab. There is an extra part, which may be of either sign, and which is presumably a rate of gain or loss of energy of the eddies. To put these things in the notation of the former paper we have the diffusion equation

$$\frac{\partial\theta}{\partial t}=\left(\frac{p}{p_0}\right)^{0\cdot29}\frac{\partial}{\partial p}\left(\xi\frac{\partial\Phi}{\partial p}\right),$$

where $\theta$ is absolute temperature, $\Phi$ potential temperature,

* L. F. Richardson, London Roy. Soc. Proc. A, vol xcvii. p. 354 (1920).

namely the temperature which the air would attain if brought adiabatically to a standard pressure $p_i$, and $\xi$ is the turbulivity.

Instead of making a partial integration, as formerly, let us apply the formula for the differentiation of a product to the second member. Then after multiplying by $\gamma_p . \delta p/g$ we have

$$\gamma_v \frac{\delta p}{g}\frac{\partial \theta}{\partial t} + b\frac{\delta p}{g}\frac{\partial \theta}{\partial t} = \gamma_p \frac{\delta p}{g}\frac{\partial}{\partial p}\left\{\left(\frac{p}{p_i}\right)^{0\cdot 29}\xi\frac{\partial \Phi}{\partial p}\right\} - \frac{\gamma_p \delta p}{g}\frac{\partial \Phi}{\partial p}\frac{0\cdot 29}{p_i}p^{-0\cdot 71}.$$

The first term in this equation is the rate of increase of thermal energy inside the slab, the second term is the associated rate of increase of gravitational energy in the whole of the atmosphere above the slab. In the second member the first term has been shown elsewhere to be the difference between the fluxes of heat over the two faces of the slab. Consequently the last term must be the rate, per area and per time, of a supply of energy to the eddies. After various uninteresting transformations it is shown that this supply to the eddies amounts per volume to

$$\frac{cg}{\theta}\left\{\frac{\partial \theta}{\partial h} - \frac{\partial \theta_A}{\partial h}\right\} = \left(\frac{\partial e'}{\partial t}\right)_1 \text{ say,}$$

where $\partial\theta_A/\partial h$ is the adiabatic up-gradient of temperature and $c$ is the "eddy-conductivity" defined by

$$\frac{\bar{D}\Phi}{\bar{D}t} = \frac{1}{\rho}\frac{\partial}{\partial h}\left(c\frac{\partial \Phi}{\partial h}\right),$$

where $\rho$ is the density and $\bar{D}$ a differentiator following the *mean* motion.

The other causes of supply of energy to and from eddies must now be named. As Osborne Reynolds showed, eddy energy is produced by shearing of the mean-wind at a rate per volume equal to

$$\left(\frac{\partial e'}{\partial t}\right)_2 = \mu_{XH}\left(\frac{\partial \bar{v}_X}{\partial h}\right)^2 + \mu_{YH}\left(\frac{\partial v_Y}{\partial h}\right)^2,$$

where $\mu_{XH}$, $\mu_{YH}$ are eddy-viscosities.

Next it is obvious that molecular viscosity $\mu_c$ damps out eddies. In the former paper it is shown that in some

instances, taken as typical, $(\partial e'/\partial t)_3 = -\frac{2\mu_c\pi^2}{\rho l^2}e'$, where $l$ is a conventional linear dimension of an eddy and $\rho$ is the air-density.

Lastly it seems likely that eddy-energy, produced in a thin layer, surrounded by non-turbulent air, will not all remain in the layer, but that some of it will disperse as eddy energy. A rough experiment to illustrate this effect was made in a domestic bath full of still water. A fountain pen-filler, 1·1 cm. in diameter, was drawn in a straight line through the top of the water at a speed of 31 cm. sec.$^{-1}$ and was squeezed meanwhile so as to leave a wake of inky eddies. These gradually spread out into the still water. The extreme width $s$ of the ink-trail was observed in the middle of the bath every 20 seconds. It was found that $s^{2\cdot4}$ was proportional to $t$ the time from the formation of the trail. The standard deviation of the ink from the middle of the lamina must have been of the order of $s/3$, so that, if we had to do with the spreading of a lamina in a medium of uniform diffusivity, and this is a very rash hypothesis, $s^2$ would be proportional to $t$, and the diffusivity would be of the order of $(s/3)^2/2t$. In the present instance this diffusivity came very roughly to $\frac{1}{3}$ cm.$^2$sec.$^{-1}$. Such a diffusivity, if it exists, probably increases both with the size of eddies and with the speed of their particles. All we can do at present in connexion with turbulence among trees is to neglect this effect, and to notice whether we are thereby led to false conclusions. As a reminder of its existence let us denote this part of the rate of supply per volume of energy to eddies by $(\partial e'/\partial t)_4$.

Collecting terms

$$\frac{\partial e'}{\partial t} = \left(\frac{\partial e'}{\partial t}\right) + \left(\frac{\partial e'}{\partial t}\right)_2 + \left(\frac{\partial e'}{\partial t}\right)_3 + \left(\frac{\partial e'}{\partial t}\right)_4$$

$$= \frac{cg}{\theta}\left\{\frac{\partial\theta}{\partial h} - \frac{\partial\theta_A}{\partial h}\right\} + \mu_{XH}\left(\frac{\partial v_X}{\partial h}\right)^2 + \mu_{YH}\left(\frac{\partial v_Y}{\partial h}\right)^2 - \frac{2\mu_c\pi^2}{\rho l^2}e' + \left(\frac{\partial e'}{\partial t}\right)_4 . \quad . \quad (1)$$

G. I. Taylor * and Schmidt † have given reasons for believing that $c = \mu_{XH} = \mu_{YH}$. If we neglect the dispersal of eddies, then as the term $-2\mu_c\pi^2e'/(\rho l^2)$ due to molecular

* G. I. Taylor, "Eddy Motion in the Atmosphere," Phil. Trans. Roy. Soc. A. ccxv. (1914).

† Schmidt, *Akad. Wiss. Wien*, 1917.

viscosity tends to make $\partial e'/\partial t$ negative until $e'$ vanishes, it follows that eddies will just die away if

$$\left(\frac{\partial v_X}{\partial h}\right)^2+\left(\frac{\partial v_Y}{\partial h}\right)^2=\frac{g}{\theta}\left\{\frac{\partial\theta}{\partial h}-\frac{\partial\theta_A}{\partial h}\right\}. \quad . \quad . \quad . \quad (2)$$

C. K. M. Douglas * has recently shown that this criterion (2) will explain the scarcity of sudden inversions of temperature in the upper-air.

When we compare formula (2) with observation we have to replace infinitesimals by finite differences, and it must be admitted that the differences of 14 and 24 metres which have been employed are rather too large for this region of steep non-linear gradients; and further that it would have been better, had it been possible, to have made the two ranges coincident, to have measured the turbulence in the middle instead of at the top, to have recorded the direction of the lower as well as of the upper wind, and to have had the trees quite uniform in size, kind, and arrangement. However, in spite of all these imperfections, we shall find something of interest.

If $v_{26}$ and $v_2$ are the wind speeds in metres per second at 26 and 2 metres above ground, and if the junction at 18·3 metres is $t°$C. warmer than the junction at 4·4 metres, then formula (2) takes the approximate form

$$\tfrac{1}{4}(v_{26}-v_2)^2=1{\cdot}43(t+0{\cdot}137). \quad . \quad . \quad . \quad . \quad (3)$$

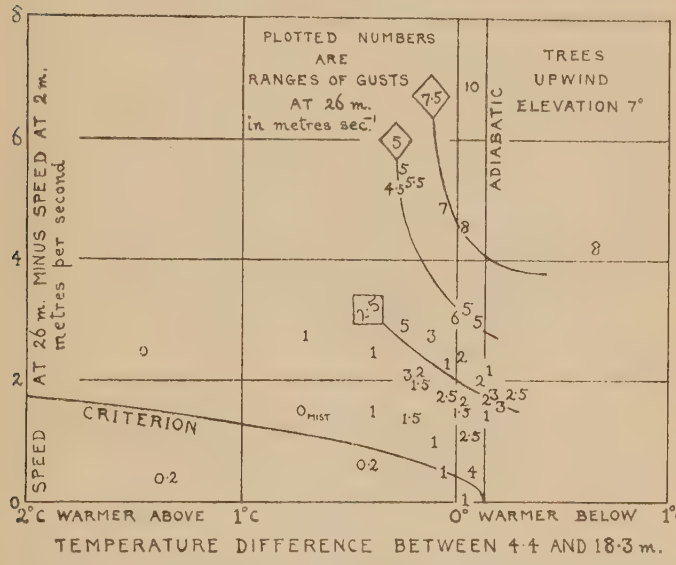
The approximation consists in neglecting the variation of wind direction between the two levels. I doubt if it ever exceeded 20° and consequently it was not important.

To compare the observations with formula (3) a diagram (fig. 4) has been constructed having vertical temperature-difference for one coordinate and vertical wind-difference for the other. Each observation is represented by a point on the diagram, and the range of gusts on that occasion is written on the diagram as a number having its centre at the point in question. The curve represented by equation (3) is plotted on the diagram, and it is found that it does really lie near the boundary of the region of no gusts. This is surprising when we consider that among trees the mean wind does not flow in straight lines but bends round every leaf and twig, so that we might reasonably expect that friction with the trees would generate eddy-energy at a considerable rate, in addition to the part represented by the first member of (2). The present observations therefore

* Quart. Journ. Roy. Meteor. Soc. Oct. 1924, p. 359.

suggest that the two members of equation (2) are derived from the most important parts of the rate of generation of eddy-energy, even when there are trees close below the level where the turbulence is measured.

Fig. 4.


PLOTTED NUMBERS ARE RANGES OF GUSTS AT 26 m. in metres sec.⁻¹
TREES UPWIND ELEVATION 7°
ADIABATIC
SPEED AT 26 m. MINUS SPEED AT 2 m. metres per second
CRITERION
SPEED
2°C WARMER ABOVE 1°C
0° WARMER BELOW 1°C
TEMPERATURE DIFFERENCE BETWEEN 4·4 AND 18·3 m.

## § 5. *Degrees of Turbulence.*

Most of the observations lie far away from the criterion in the gusty region. Isopleths of range of gusts have been sketched to represent the observations. In this region the eddy-energy would be increasing if the second member of (2) represented the only cause acting to decrease it There is of course another cause; the breaking up of large eddies into smaller ones which in turn are damped out by molecular viscosity. Apparently this cause is important when turbulence is great and yet negligible near the criterion.

## § 6. *Personal.*

These observations at Benson Observatory (Long. 1° 6′ W., Lat. 51° 37′ N.) were begun as an official duty on the field of Mr. W. H. Dines, by whom they were helped on. The screens were made by Mr. B. C. Lewis. The results are communicated by permission of the Director of the Meteorological Office.

§ 7. *Summary.*

(1) Statically unstable vertical gradients of temperature were common between heights of 4·4 m. and 18·3 m. in the middle of the day. See fig. 2.

(2) The "gustiness" at 26 m. increased as the layer below became less statically stable. But after the layer had passed the labile adiabatic condition, the gustiness did not further increase, although the layer below was definitely unstable. See fig. 3.

(3) The criterion of just-no-turbulence, published by the author in 1920 *, is in rough agreement with the observed facts; in spite of the friction of the air on the trees. See fig. 4. But the observations do not give a definite answer to the question as to whether a factor of 2 does come in, as it would according to calculations by G. I. Taylor if the eddies developed from waves in shearing laminæ.

(4) Isopleths of the range of gusts are drawn on fig. 4 for the region far from the criterion.

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## VIII. *Periodic Precipitation in Dilute Gelatine Gels.*

*By* G. W. Scott Blair, *B.A., Trinity College, Oxford* †.

[Plate III.]

WHEN the solution of one salt is allowed to diffuse into a jelly containing another salt with which the first interacts to give a sparingly soluble compound, the precipitate is formed in many cases in separate rings instead of in a continuous band.

This phenomenon, which was discovered by Liesegang, has been shown to be the cause of many periodic formations in nature, and is thus of considerable importance.

*The effect of the conditions of the gel on the phenomenon.*—The experiments of Dreaper (*Koll. Zeitschr.* xiv. p. 163, 1914), Kauer (J. A. C. S. xliv. p. 951, 1922), and Chapin (see Holmes, J. A. C. S. xl. p. 1189, 1918) have shown that the phenomenon can take place in the absence of gels; but there can be no doubt that the nature and concentration of the gelatine exert a profound influence. [See Liesegang, *Koll. Zeitschr.* ii. p. 73; Hausmann, *Zeit. für Anorg. Chem.* xl. p. 110, 1904; Köhler, *Koll. Zeitschr.* xix. p. 65, 1916; Bradford, Biochem. Journ. x. p. 169, 1916, etc.]

* Roy. Soc. Proc. A. xcvii. p. 354 (1920).

† Communicated by the Author.

Liesegang points out in the paper quoted above that the age of the gel may modify ring formation, but he does not arrive at any very definite conclusions on the subject.

It seemed that further work should be carried out on the effect of the age of the gel on the formation of rings. Experiments described in this paper and illustrated in Pl. III. show the rather remarkable effect of age. In these experiments silver nitrate diffused into potassium bichromate in dilute gelatine. The gels were made up under standard conditions—10 per cent. silver nitrate, 0·019 per cent. potassium bichromate, and 0·5 per cent. and later 1 per cent. gelatine being used. It will be seen that in a given tube in which ring formation occurs the number of rings formed is quite definite, and the distance from the surface of the last ring to be formed is quite sharp. This distance is called "$a_\infty$." It will also be seen that $a_\infty$ increases regularly with the age of the gel from zero-age to about ten days. With gels older than this, ring formation ceases to take place at all (traces of irregular vertical striations are, however, produced in the tube instead of regular horizontal bands). Then, with gels of very much greater age, the power of ring formation is found to be recovered once more (not shown in figure). The "*Ageing series*" exhibited by gels of age varying from 0 to 10 days is clearly shown in the figure.

A similar progressive increase of $a_\infty$ was not found with other salt pairs; but with silver nitrate and potassium chromate, the second part of the phenomenon (failure to produce rings after a certain limit of age) made its appearance. In the case of the bichromate, this age-limit was found to increase on increasing the gelatine concentration (see Table I.).

Special experiments were made to show that the effect of ageing was not due to any of the following causes :—

(1) Changes in the rate of diffusion of the ions through the gelatine, since not only did the electrical conductivity remain unchanged as the gel aged, but also the observed rate of diffusion of various coloured salts and dyes through the gelatine was found to be independent of its age.

(2) Changes in the H-ion concentration of the system.

(3) Changes in bichromate concentration due to chemical interaction with the gelatine.

Some change in the structure of the gel is therefore indicated as the cause.

TABLE I.

Effect of gelatine concentration on age-limits for ring formation.

| Percentage Gelatine. | Age-limit to ring formation. |
|---|---|
| 0·5 ..... | 6–13 days. |
| 1 ..... | 23–28 hrs. |
| 2 ..... | 10–46 hrs. |
| 5 ..... ..... | 10–23 hrs. |

*Note.*—Throughout the course of these experiments, the room-temperature at which they were carried out varied over about three or four degrees.

*The effect of light on the phenomenon.*—That light has a profound influence on the phenomenon has been shown by Davies (J. A. C. S. 1923, p. 2261), Kuester (*Koll. Zeitschr.* xiii. p. 192), Hatschek (Proc. Roy. Soc. 1921, p. 496), and Sen & Dhar (*Koll. Zeitschr.* xxxiv. p. 270, 1924). The first of these authors found that blue light was more effective than red in its action in the case of the reduction of gold chloride in silicic acid gels in which coloured bands are produced. The second found that ring formation became impossible in complete absence of light (though, since he used gels of varying ages, any generalization of his results seems impossible); and the third that in a few special cases abnormal ring formation takes place under the influence of light. The work of Sen & Dhar extends and verifies that of the above-mentioned workers.

Work has been done by the author on the effect of light of different wave-lengths on the production of silver-bichromate rings in gelatine, and it has been shown that:—

(1) Either intense white light ($\frac{1}{2}$ watt lamp 6 in. range) or complete darkness cause irregular ring formation, and a decrease in the value of $a_\infty$.

(2) The diminution of $a_\infty$ is greater in the case of blue light than in the case of red light of the same intensity.

(3) The action of any kind of visible light is irreversible, in the sense that, after reliquefaction and resetting of the gelatine, the anomalous formation persists.

(4) Ultra-violet light from a mercury-vapour quartz lamp makes the formation of rings in a gel exposed to it for even a few hours quite impossible.

(5) Unlike the effect of visible light, which is permanent,

that of ultra-violet light does not persist if the gel is reliquefied and reset. This remarkable difference seems to be fundamental.

If the gel is exposed to the action of both types of light simultaneously, reliquefaction and resetting reverses the action of the ultra-violet but not of the visible light. Ring formation is then exactly as if the gel had been exposed to visible light only. These curious phenomena were observed repeatedly, and not merely on a single occasion.

Incidentally it may be remarked that the action of ultra-violet light does not alter the rate of setting of the gelatine.

*The cause of the phenomenon.*—It is clear from the above that outside factors, such as the physical conditions of the gel and the action of light, have a profound effect on the phenomenon ; but, as pointed out already, the periodic nature of the precipitation is not itself essentially dependent on the presence of the gel.

The only theory as yet put forward in the literature to explain rhythmic banding which seems to be supported by strong positive evidence [see Morce & Pierce (*Zeit. für Phys. Chem.* xlv. p. 589, 1903), Jablczynski (*Bull. Soc. Chim.* 1923, p. 1594, etc.)] is that of Ostwald, who postulates that the resultant salt is retained in supersaturated solution until a sharp limit of concentration (the metastable limit) is reached, after which a precipitate is formed. Back-diffusion then impoverishes the zone immediately below the precipitate to such an extent that the metastable limit is no longer exceeded, and a clear space ensues. This process is repeated.

Although Ostwald's theory is undoubtedly the correct explanation of the phenomenon in its essentials, it requires considerable modification in detail. Three serious objections to it, as it stands, arise :—

(1) Hatschek (*Koll. Zeitschr.* x. p. 124, 1912) has found that previous sowing of the gel with nuclei of the precipitate does not relieve supersaturation, but that ring formation still takes place. The experiments, which were repeated by the author, consisted in adding a little lead nitrate to an agar sol containing potassium iodide before the sol set, so that on setting, the gel was filled with small particles of lead iodide. On allowing a solution of lead nitrate to diffuse into this gel, rings were formed quite independently of the presence of the lead-iodide nuclei throughout the gel.

(2) The fact that the condition of the gel has a profound effect on ring formation is quite unexplained.

(3) On the simple Ostwald theory, the values of $a/\sqrt{t}$

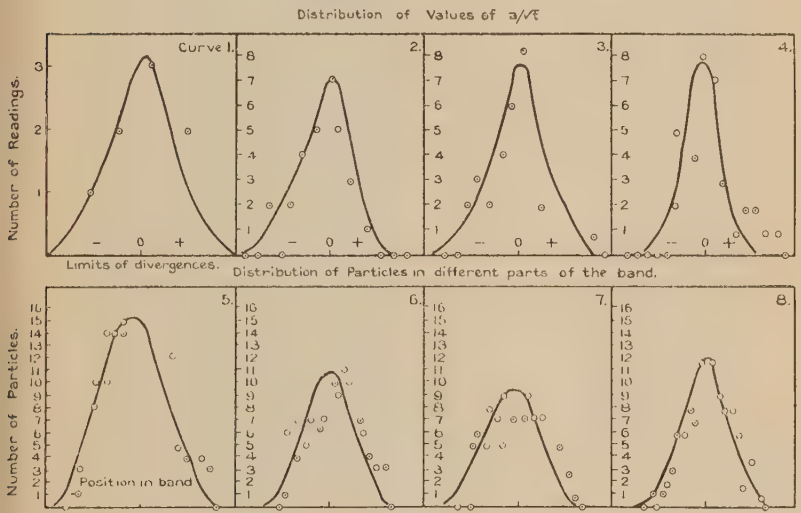
(where $a$ is the distance of any ring from the origin, and $t$ is the time interval from the addition of silver nitrate to the tube until the ring is formed) should be constant. Morse & Pierce found that this was approximately the case (*loc. cit.*); but the author, by observing carefully the exact time of formation of the rings, has shown that the values fluctuate about a mean constant value to a degree unexplainable on the hypothesis of mere errors in measurement (see p. 96).

The following modification of Ostwald's theory, which seems to avoid these three difficulties, has been suggested to me by Mr. C. N. Hinshelwood. There is no absolutely sharp metastable limit, but only a region of concentration where precipitation becomes very probable. In the immediate neighbourhood of the ordinary solubility limit precipitation is very unlikely, in the sense that it would only occur on the average after a very long time. This probability of crystallization increases very rapidly when we get to a certain region of concentration, but there is no sharp limit above which precipitation does occur and below which it does not occur (*cf.* Hinshelwood & Hartley, Phil. Mag. xliii. p. 78, 1922). The second suggestion is that the micelles of the gel itself may act as crystallizing centres for the formation of precipitate, just as colloidal particles of the right size were found to provoke the crystallization of supercooled liquids in the paper referred to.

The first part of this suggestion has been tested by the author in two ways, and in both cases the evidence obtained seemed strongly to favour the above explanation. In the first place, as mentioned above, the exact times of precipitation of rings in about 8 tubes simultaneously were noted, and values of $a/\sqrt{t}$ were worked out. The curves obtained by plotting the number of values of $a/\sqrt{t}$ within various fixed distances from the mean value are given, and give a curve of the usual probability type (see curves 1–4). It can be easily shown from known errors in $a$ that the discrepancies in $a/\sqrt{t}$ are far too great to be due merely to experimental error in measurements of $a$ and $t$ (see p. 96).

In the second place, the idea of a metastable region as above described should manifest itself in a falling off of the number of particles in the rings from the centre in accordance with the ordinary chance distribution. Certain plates of Hatschek and Liesegang seem to show such a falling off (*e. g.* Soc. of Chem. Ind. Proc. xxx. p. 256, 1911), though neither author comments on it; but it has been

found possible in this work to get much more definite information on this score by producing rings so wide, and containing so few particles of precipitate, that the number in different parts of the ring could be counted under a high-power microscope. The resulting curves (curves 5–8) are of a form in complete accordance with the probability theory. It is not surprising, on this theory, that the conditions of the gel should affect ring formation. Whether we consider the formation of a visible precipitate as spontaneous crystallization, or as the coagulation of already formed colloidal silver bichromate, is immaterial. In the former




case it is possible, as stated above, that the micelles of the gelatine act as crystallizing nuclei when of the appropriate size and when the appropriate concentration is reached: the variation of effectiveness with age depends on the supply of micelles of appropriate size, since ageing is known to increase the size of the micelles. (This increases $a_\infty$, whereas the action of light, which is shown to diminish the micellar size, diminishes the value of $a_\infty$.) If we take the second view, the condition of the gelatine would certainly influence the phenomenon profoundly, in so far as it would exert a greater or less protective effect on the coalescing particles. But further, the idea that the micelles of the gel of appropriate size are themselves crystallizing centres, avoids the difficulty of the inoculation experiments.

Proof that the discrepancies in values of $a/\sqrt{t}$ are not due to reading errors :—

*Data* :

Measurements gave the following :—

| | | | |
|---|---|---|---|
| Mean value of $a/\sqrt{t}$ for all readings | $(y)$ ..... | 0·00908 | c.g.s. units. |
| Mean value of $a$ ,, ,, | $(a)$ ..... | 0·4897 | |
| Mean value of $t$ ,, ,, | $(t)$ ..... | 3063 | |
| Mean error in reading $a$ (determined) | $(\Delta a)$ ... | 0·011 | |
| Mean discrepancy in $a/\sqrt{t}$ ..... | $(\Delta y)$ ... | 0·00049 | |

From these figures we can calculate what the error in reading $t$ should be, if we assume that the discrepancies in $a/\sqrt{t}$ are due entirely to errors in reading.

From the equation

$$\log y = \log a - \tfrac{1}{2}\log t$$

we get

$$1/y \,.\, \Delta y = 1/a \,.\, \Delta a - 1/2t \,.\, \Delta t,$$

or

$$\Delta t = 2t(\Delta a/a - \Delta y/y).$$

Assuming the errors in $a$ and $y$ to be

| (1) Of the same sign : | (2) Of opposite sign : |
|---|---|
| $t = 6126(0{\cdot}0011/0{\cdot}4897 + 0{\cdot}00049/9{\cdot}00908)$ | $t = 6126(0{\cdot}0011/0{\cdot}4897 - 0{\cdot}00049/0{\cdot}00908)$ |
| $= 6126 \times 0{\cdot}05621$ | $= 6126 \times 0{\cdot}05171$ |
| = 5 min. 44 sec. | = 5 min. 17 sec. |

In either case this is absurd as a mean error in reading $t$, which can be read easily to the nearest second at the outside, so that the original assumption is unjustified, and we are brought to the conclusion that the discrepancies in $a/\sqrt{t}$ are not due merely to errors in reading $a$ and $t$.

## *Conclusions.*

It is shown in this paper that the age of gels profoundly influences Liesegang ring formation in them.

The effect of light of various wave-lengths has been investigated. A fundamental difference between the effects of ultra-violet light and of visible light has been found.

A new modification of the Ostwald theory of the phenomenon is discussed, and evidence in its favour put forward.

In conclusion, I should like to express my gratitude to Mr. C. N. Hinshelwood, who has directed this research throughout, and whose suggestions and criticisms have made possible the production of this paper.

Trinity College, Oxford,
July 18th, 1924.

1 2 3 4 5 6 7

Ages.

| Tube. | Age. |
|---|---|
| 1 ..... | Zero. |
| 2 ..... | $\frac{1}{2}$ hr. |
| 3 ..... | $3\frac{1}{2}$ hrs. |
| 4 ..... | $18\frac{1}{2}$ hrs. |
| 5 ..... | $42\frac{1}{2}$ hrs. |
| 6 ..... | 6 days. |
| 7 ..... | 13 days. |

## IX. *On the Forces at the Ends of a Vibrating String.* By W. H. George, *M.Sc., University College, Nottingham* *.

*Summary.*—The paper deals with the magnitude and frequency of those forces at the ends of a vibrating string which are due to the tension of the string. The influence upon the length of the string of its change of shape when vibrating is calculated, and the result is used to determine the increase in tension. Incidentally, the maximum tension for a given amplitude of vibration is determined theoretically, and numerical values are calculated and compared with the tensions used for pianoforte strings where high tensions are shown to be of great importance. The effect of the inclination of the vibrating string upon the longitudinal and normal forces at the ends of the string is illustrated graphically. Calculations of the resultant longitudinal and normal components for two steel wires of different diameter are given and show the importance of the longitudinal component which is of double frequency. Experimental work by Barton, Raman, Berry, and others dealing with actual bridges and showing the presence of the double frequency vibrations in the longitudinal motions of the bridges, is quoted. Finally, an anomalous excitation of the $n$th partial by striking a string at $1/n$th of its length from the nearer bridge is discussed.

IN treating of the theory of the transverse vibrations of strings, the string is usually taken to be a perfectly uniform and flexible filament of solid matter stretched between two fixed points, and the tension is assumed to be constant throughout the motion. The string used in experimental acoustics approximates closely to this ideal, and the small deviations from the ideal have been studied theoretically and experimentally by various workers. It would appear, however, that the yielding of the supports is of special importance and is worthy of still further study. It is well known that the aerial vibrations directly set up by a vibrating string are almost if not quite inaudible, and that the audible vibrations are generated by the motion of the larger surfaces to which the ends of the string are attached. If, then, these ends were fixed, the vibrating string would be inaudible. Motion of an end of the string is also tacitly assumed when the tuning of the string is tested by pressing the stem of a vibrating tuning-fork upon one of the ends.

Rayleigh ['Sound', i., § 135] has considered in what manner the vibrations of a string are liable to be modified by a yielding of the points of attachment. Assuming the

* Communicated by Prof. E. H. Barton, F.R.S.

system frictionless and symmetrical with reference to the centre of the string, he finds (i.) that if each end be attached to a particle of mass (M) there is a rise in pitch, the rise being greater the lower the component tone, and (ii.) that if each end is urged by a spring ($\mu$) towards the position of equilibrium then the pitch of the whole series of partials is slightly lowered. In an actual case, where both mass and spring factors are present, the raising or lowering of pitch will depend on whether the natural note of the mass (M), urged by the spring ($\mu$), is lower or higher than that of the component vibration in question. Experimental work bearing on the motion of the ends will be referred to later, and we will now consider the forces which cause this motion. We have (i.) the change in length and consequent change in tension of the string due to its change of shape, and (ii.) the change in direction at the ends of the string of the force due to its tension.

## Extension of the Vibrating String.

Since the increase of tension can be calculated from the increase in length, we will first calculate the increase in length of the string due to its varying shape. The shape of the string at any instant $t$ is given by the equation

$$y = \sum_{s=1}^{s=\infty} \sin\frac{s\pi x}{l_0}\left(\mathrm{A}_s\cos\frac{s\pi at}{l_0} + \mathrm{B}_s\sin\frac{s\pi at}{l_0}\right), \quad . \quad (1)$$

giving the displacement $y$ of any point $x$, the origin of co-ordinates being at one end of the string, which is here assumed to be perfectly uniform and flexible, to be held fast at both its ends distant $l_0$ apart, and to vibrate under uniform tension in one plane only. The notation is otherwise as in Rayleigh's 'Sound,' i. § 124. Since the length ($l$) of the string is given by

$$l = \int_0^{l_0} \sqrt{1+(dy/dx)^2}\,.\,dx,$$

we have:

$$l = \int_0^{l_0}\left[1+\sum_{s=1}^{s=\infty}\frac{s^2\pi^2}{l_0^2}\cos^2\frac{s\pi x}{l_0}\left(\mathrm{A}_s\cos\frac{s\pi at}{l_0}+\mathrm{B}_s\sin\frac{s\pi at}{l_0}\right)^2\right]^{\frac{1}{2}}.dx. \quad . \quad . \quad . \quad (2)$$

*String Sounding Fundamental.*—To calculate the order of magnitude of the extension we must take some special case, since the values of the coefficients $\mathrm{A}_s$ and $\mathrm{B}_s$ depend on the

initial values of $(y)$ and $(dy/dt)$. The simplest case, from which all others can be deduced, is that of the string sounding its fundamental only. If the maximum displacement of the middle point of the string be A, then equation (1) reduces to

$$y = \mathrm{A}\sin\frac{\pi x}{l_0}, \qquad \ldots\ldots\ldots \quad (1\,a)$$

and

$$l = \int_0^{l_0} \sqrt{1 + \frac{\mathrm{A}^2\pi^2}{l_0^2}\cos^2\frac{\pi x}{l_0}}\,.\,dx.$$

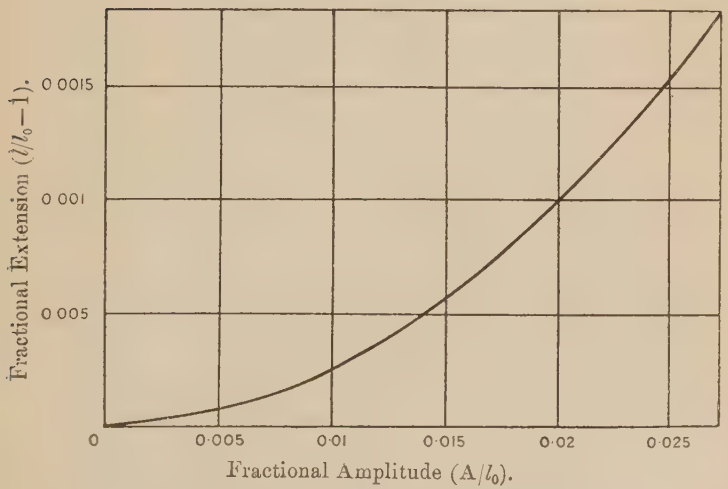
This can be rewritten:

$$l = \sqrt{1+\mathrm{K}^2}\int_0^{l_0}\sqrt{1 - \frac{\mathrm{K}^2}{1+\mathrm{K}^2}\sin^2\frac{\pi x}{l_0}}\,.\,dx,$$

where $\mathrm{K} = \mathrm{A}\pi/l_0$. Changing the variable by putting $\phi = \pi x/l_0$, we have:

$$l = \frac{2l_0}{\pi}\sqrt{1+\mathrm{K}^2}\int_0^{\frac{\pi}{2}}\sqrt{1 - \frac{\mathrm{K}^2}{1+\mathrm{K}^2}\sin^2\phi}\,.\,d\phi. \quad . \quad (2\,a)$$

Fig. 1.




From a table of the complete elliptic integral of the second kind numerical values of the integral in equation $(2\,a)$ were obtained and were used to plot the graph shown in fig. 1. The fractional extension $(l/l_0-1)$ produced in a freely

vibrating string due to its changing shape when sounding the fundamental tone only, is there plotted against the fractional amplitude $(A/l_0)$. The assumptions used in the calculation are stated after equation (1).

If Y = Young's Modulus for the material of the wire, and $r$ is the radius of the wire, the increase in tension $\delta T$ is given by

$$\delta T = \pi r^2 Y(l/l_0 - 1).$$

On account of the high speed of propagation of longitudinal disturbances the variations of tension with position along the string may be neglected. It will be noted that (i.) the increase in tension is not directly proportional to the amplitude, and (ii.) the longitudinal component (along the axis of $x$) varies with a *frequency double that of the string's vibrations*. Perhaps the best experimental illustration of this second result is shown in a paper, "Photographs of Vibration Curves," by C. V. Raman (Phil. Mag. xxi. pp. 615–618, May 1911). The string used to obtain the photographs there shown had one end attached *normally* to a sounding-board, and the other end attached to a rigid support. When the string vibrated in a horizontal plane it was found that the motion of the sounding-board had a frequency strictly double that of the vibration of the string, no matter how the string had been excited. When, however, the vibrations took place in a vertical plane the fundamental appeared, due to the fact that the equilibrium position of the string was then a catenary and the change in length about the equilibrium position unsymmetrical. In the simultaneous vibration-curves of string and bridge photographically obtained from a monochord by E. H. Barton and J. Penzer (Phil. Mag. xiii. pp. 446–452, April 1907), the vibrations of double frequency are strongly present in the longitudinal motions of the bridge, although the fundamental can be detected in all the figures (see Pl. XI. figs. 7–13 and 27–39, also Pl. XII. figs. 40–47). The longitudinal displacement of the bridge was found to be about seventeen times the amplitude of the transverse motion. The motions of the violin bridge have been studied by E. H. Barton and T. F. Ebblewhite (Phil. Mag. xx. pp. 456–464, Sept. 1910), and were found to be still more complicated than the motions of the monochord bridges. The vibrations of double frequencies were sometimes present in the longitudinal motions, but other figures are shown in which a vibration much slower than that of the string is present (see, for example, Pl. IX. figs. 31, 37, 50, 60, 66, and 71). In a paper "On the

Pianoforte Bridges," by G. H. Berry (Phil. Mag. xxii. pp. 113–118, July 1911), two photographic records are given, fig. 3 showing the transverse vibrations of the "free" bridge, and fig. 4 the longitudinal vibrations; the latter are of strictly double the frequency of the former. It is of interest to note that the string was here stretched vertically. We shall return later to the consideration of the frequencies of the string's terminal forces.

*Maximum Amplitudes and Tensions.*—In order that a stretched string may behave elastically, it is necessary that its tension shall not exceed a certain value which is a constant for the material of the wire, although for a particular wire it depends also on the previous history of the wire. Hence, there is a theoretical limit to the amplitude of the vibrations of a string stretched with a given tension between firm supports. It is believed that the following considerations show the practical importance of a calculation of this theoretical limit. In the study of the struck string, it is found that the time of contact of the hammer and the string is usually some fraction of the period of the fundamental tone of the string. Hence, the whole of the energy communicated to the system has to be transferred from the hammer to the string in an exceedingly short time, which may well be one five-hundredth of a second, or less. Thinking now of the acoustics of the pianoforte, it is clear that either the loudness of the the sound or its duration, or both, will in a given instrument depend on the energy communicated to the system. The greater the tension, the greater the work done by the hammer in producing a given displacement of the string, and consequently for a given amplitude of vibration more energy can be given to the system. The length and linear density of the string can be adjusted to leave the pitch of the string unaltered without affecting the validity of these considerations. Hence we see the importance in pianoforte construction of using tensions as high as possible. Whether the increased supply of energy appears chiefly in louder sounds or in sounds of greater duration, will depend upon the construction of the sound-board and upon other considerations, studies of which by the present writer it is hoped to publish in later papers [see also Phil. Mag. xlviii. p. 42, July 1924]. It is of interest to note that in instruments involving the struck string the greatest difficulty has always been the communication of sufficient energy in a single blow to yield a sound sufficiently loud and sustained. The material used for the strings has been gut, silk

prepared by a special process, latten, gold, silver, brass, iron, and, finally, steels of increasing tensile strength. In 1867, the breaking stress for carefully prepared steel wire of 0·77 mm. diameter was 226 lb. wt.; in 1893 this figure had been increased to 325 lb. wt. The increase in the tensions of the strings made possible by this increase in tensile strength resulted in a corresponding increase in intensity and duration of the tones. "It is not too much to say that the improvements in the tenacity and elasticity of steel wire made during the last century have rendered possible the modern piano," Wolfenden, 'Treatise on the Art of Pianoforte Construction,' 1916, p. 6.

If T be the original tension of the string, $\delta T$ the maximum increase of tension due to the change of shape of the string when vibrating, and $T'$ the tension when the elastic limit is reached, then we may write for a maximum $T = T' - \delta T$. The value of T so found will then be the desirable tension for the particular wire in order that the energy of a vibration of given fractional amplitude may be a maximum. In Table I. are shown values of T for certain cases of practical

Table I.

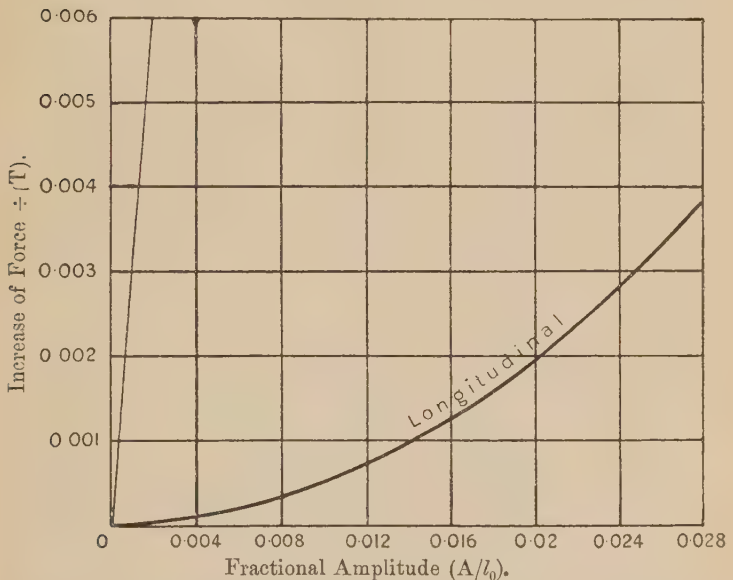
| Frequency. | $l_0$. | $r$. | $\delta T$. | $T+\delta T$. | Probable breaking tension ($\fallingdotseq T'$). | $T'-(T+\delta T)$. |
|---|---|---|---|---|---|---|
| per sec. | cm. | cm. | dynes. | dynes. | dynes. | |
| 4138·4 | 5·4 | 0·04 | $2·61\times10^6$ | $79·61\times10^6$ | $122·2\times10^6$ | $42·6\times10^6$ |
| 2069·9 | 10·2 | 0·0425 | 2·95 | 79·95 | 136·1 | 56·1 |
| 1034·6 | 19·25 | 0·045 | 3·31 | 80·31 | 147·7 | 67·4 |
| 517·3 | 36·4 | 0·0475 | 3·68 | 80·68 | 162·8 | 82·1 |
| 258·6 | 68·8 | 0·05 | 4·08 | 81·08 | 170·8 | 89·7 |
| 129·3 | 131·0 | 0·0525 | 4·49 | 81·49 | 182·4 | 100·9 |

interest, calculated from data given in Wolfenden (*loc. cit.* pp. 28–29), on the assumption that $A/l_0$ shall not exceed 0·01. The writer was unable to find numerical values for $T'$ for piano wire, but since the elastic limit, the yield point, and the breaking point are in this case close together, values of the breaking stresses have been used for $T'$. It will be noted that the original tension of all the strings here considered is the same. For a consideration of the factors which determine the tensions chosen for the strings of a pianoforte, the reader is referred to Wolfenden, *loc. cit.*, especially pp. 5–23, or to the 'Dictionary of Applied Physics,' iv. pp. 464–470.

### Inclination of the String at its Ends.

It is convenient to resolve the force due to the inclination of the string at its ends along and perpendicular to the equilibrium position of the string, since the rigidness of the supports actually used in experiments is usually most conveniently considered in these two directions. If the tension of the string be $T_1$ when the inclination of the string with its equilibrium position is $\phi$, we have components $T_1 \cos \phi$ longitudinally and $T_1 \sin \phi$ normally. We may note that (i.) the longitudinal component is again of frequency double that of

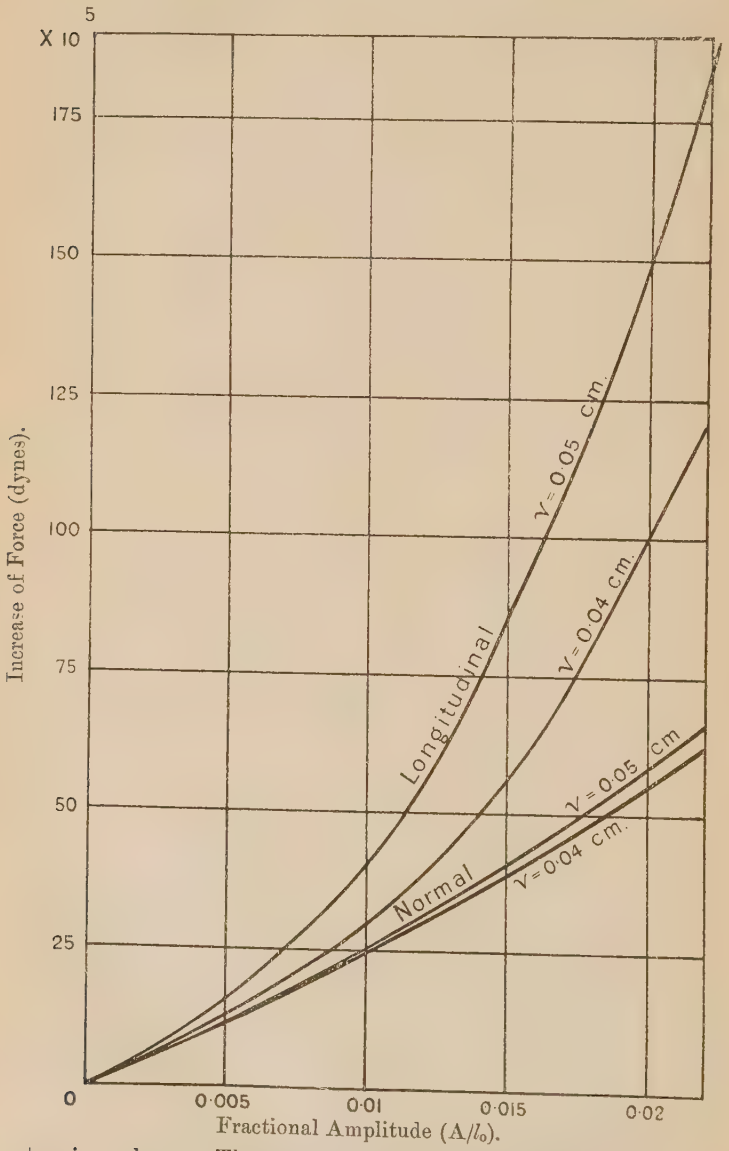
Fig. 2.




the string's vibrations, and (ii.) the inward force at the ends of the string is *decreased* when the string is displaced, this acting in opposition to the increased force due to the extension of the string. It would appear that the longitudinal component was much greater than the normal, but it should be remembered that when the string is at rest the longitudinal component is T, whilst the normal component is zero. We are, therefore, concerned rather with the component $(T_1 \cos \phi - T)$ which represents the *change* in the longitudinal force when the string is vibrating. In the lower curve of fig. 2, this component, divided by T, is plotted against the fractional amplitude $(A/l_0)$. In the upper curve the (normal force) ÷ (T) is similarly plotted. Since in fig. 2 the tension

has been assumed constant, we see the influence of the inclination alone, whilst in fig. 1 we see the influence of the

Fig 3.




extension alone. The two influences are combined in fig. 3, which deals with two actual numerical examples, those of

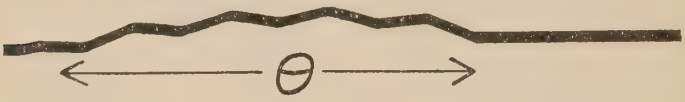
steel wires, for which $Y = 20 \times 10^{11}$ dynes per sq. cm., $T = 77 \times 10^6$ dynes, and $r = 0{\cdot}05$ cm. and $0{\cdot}04$ cm. respectively. A length 68·8 cm. of the string for which $r = 0{\cdot}05$ cm. would give a fundamental tone of frequency 258·6 per sec. The two upper curves deal with the longitudinal and the two lower with the normal components respectively. The importance of the longitudinal component of double frequency is clearly shown, but it should be noted that the audible vibrations do not usually proceed directly from the bridges, but from the sound-board to which the bridges are attached. The maintenance of a vibration by a force of double frequency, as, for example, in the longitudinal form of Melde's Experiment, is quite common, so that it may well happen that the longitudinal force of double frequency actually helps the maintenance of the single-frequency vibrations of the sound-board. It is hoped in a later paper to discuss the results of an experimental study of the motion of an actual sound-board, since in the accounts of the experimental work previously quoted there are not sufficient data concerning the mass and spring factors of the bridges there used, to enable one to calculate, for example, the theoretical amplitudes of vibration of the bridges.

## Struck String During Impact.

So far, we have considered only the freely vibrating string. A curious point arises in connexion with a string struck near one end when we consider the frequency of the forces exerted by the string upon the nearer bridge during the impact. At the beginning the transverse disturbance travels outwards from the struck point along each of the two portions of the string. One-half of the disturbance soon reaches the nearer bridge, is there reflected, and again reaches the hammer. At the hammer it is reflected, and so on until the hammer quits the string. Since the duration of the impact is of the order of the fundamental period of the string [*i. e.* the time ($\theta$) taken by the transverse disturbance to travel twice the length of the string], it follows that the disturbance is reflected a number of times at the nearer bridge. For instance, if the string be struck at $1/n$th of its length, and if the impact last for $\theta(n-1)/n$, then the nearer bridge receives $(n-1)$ impulses in the time $\theta(n-1)/n$. Hence we see that *if a string be struck near one end* at a point distant $1/n$th of its length from the nearer bridge, then, unless the duration of the impact be very near to the theoretical minimum, *the* $n$*th partial is encouraged at the nearer bridge.*

Fig. 4 shows a typical displacement-time curve of the initial motion of a point of the string between the hammer and the bridge. In this example the mass ratio ($m/M$) of hammer to string was unity, the length of the string was 316 cm., the struck point was distant 72 cm., and the observed point 35·2 cm. from the nearer bridge. The length along the time axis corresponding to the fundamental period ($\theta$) of the string is indicated. The rapid impulses illustrated in the figure are of frequency 4·39 times that of the fundamental. The experimental details will be found in the second paper of this series (Phil. Mag. xlviii. pp. 34–37, July 1924).

Fig. 4.




It is believed that this anomalous encouragement of the $n$th partial by striking the string at $1/n$th of its length from one end is connected with a special feature of pianoforte construction. In laboratory experiments with the sonometer, the two bridges are equally important, and the system is approximately symmetrical about the mid-point of the string. In the pianoforte the vibrating length of the string is determined by one "fixed" bridge and one "free" bridge. The fixed bridge is a projection on the massive iron framework which sustains the huge combined tension of all the strings; this bridge is not directly connected with the sound-board, and it plays but a small part in the communication of the string's vibrations to the sound-board. The free bridge is glued and screwed on to the sound-board itself, and is the more important in the communication of energy to the sound-board. Now, *the string is struck near the fixed bridge*, hence the initial rapid impulses of frequency $n/\theta$ (where $n$ may or may not be a whole number) are not directly communicated to the sound-board.

It is hoped to publish in a later paper quantitative results of a study of the manner in which the forces considered in the present paper are able to produce the aerial vibrations actually heard when a string is coupled with bridges and sound-board of known properties and simple construction.

In conclusion, the writer wishes to thank Mr. F. Underwood, B.Sc., and Mr. N. Davy, B.Sc., for their kindness respectively in verifying the mathematics and checking one or two of the numerical calculations.

University College,
Nottingham,
Sept. 1924.

X. *On the Reflexion of Waves penetrating normally into a Stratified Medium.* *By* J. A. Wilcken, *B.Sc., Ph.D., Lecturer in Electrical Engineering at Armstrong College, Newcastle-upon-Tyne* *.

THE question whether a gradual transition between two media of different refractive indices gives rise to any disintegration of an incoming wave-train, comparable to the reflexion in abrupt transition, has been the subject of much discussion. It is known that certain phenomena of polarization of reflected light are readily explained as due to a surface-film of varying index (*cf.* Maclaurin, 'Theory of Light,' Camb. 1908); and if that theory is adhered to, the possibility of a reflexion being caused by a continuous variation of properties must be admitted. It remains, however, to be examined whether a reflected wave is always set up in any variable medium, and to be determined on what characteristic qualities the intensity depends. The problem has acquired renewed importance in connexion with wireless signals of great wave-length.

The late Lord Rayleigh † has discussed in detail the problem of a vibrating string whose longitudinal density varies inversely as the square of the distance from one end, extending the application to phenomena other than those of sound, and concludes that reflexion is appreciable only when the transition is so sudden that the thickness of the variable layer is of the order of a wave-length; a later investigation ‡ deals with a medium built up of homogeneous strata of different qualities. From the standpoint of the electromagnetic theory of light, Gans § finds that within a certain range of approximation, characterized as "slow" transition and applicable in the domain of astrophysical optics, there is no reflexion; and Emden ‖ discards the reflected wave completely, the legitimacy of his arguments being disputed by Wallot ¶. An earlier controversy between Schmidt ** and Seeliger †† turns on a similar problem in solar physics.

It appears that previous writers have confined the discussion from the outset to a more or less specialized character

* Communicated by the Author.
† Proc. Math. Soc. xi. pp. 51–56 (1880).
‡ Proc. Roy. Soc. A, lxxxvi. pp. 207–226 (1912).
§ *Ann. d. Phys.* xlvii. pp. 709–736 (1915).
‖ *Sitz. Ber. d. Bayr. Akad.* p. 417 (1918).
¶ *Ann. d. Phys.* lx. pp. 734–762 (1919).
** *Phys. Zeitschr.* iv. pp. 282, 341, 453, 476 (1903); v. p. 67 (1904).
†† *Ibid.* v. p. 237 (1904).

of the transition ; and the alternative method of approaching a continuously varying medium from the consideration of one built up of finite homogeneous strata has been strongly criticized by Uller*. It might therefore be of some interest to pursue a different line of investigation, with a view to establishing a general expression for the relative intensity of a reflected wave, without making any special assumptions except as to the reversibility implied in the linear equation of motion.

To avoid the complications arising from the change of direction involved in oblique incidence which I have discussed in a previous paper †, we consider a train of waves proceeding in the direction of positive $x$ and impinging normally at the boundary, $x=0$, of a plane stratified medium, and take, as the equation of motion freed from the time-variable,

$$\frac{d^2U}{dx^2} + k^2U = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $k$ is a real continuous function of $x$,

$$k = 2\pi/\lambda = 2\pi n\mu/c, \qquad k\ {}^{x\leq 0} = k_0.$$

The solution may be written, without loss of generality,

$$U = Ae^{\phi+i\psi} + Be^{\phi-i\psi}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\phi$ and $\psi$ are real functions of $x$, satisfying the equations

$$\frac{d^2\phi}{dx^2} + \left(\frac{d\phi}{dx}\right)^2 - \left(\frac{d\psi}{dx}\right)^2 + k^2 = 0, \quad . \quad . \quad . \quad (3)$$

$$\frac{d^2\psi}{dx^2} + 2\frac{d\psi}{dx}\frac{d\phi}{dx} = 0. \quad . \quad . \quad . \quad (4)$$

In the homogeneous medium $x<0$ we have

$$U = He^{-ik_0x} + Ke^{ik_0x}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where H represents the amplitude of the incident positive wave, and K the amplitude of the reflected negative wave ; supposing the variable medium bounded on the right by $x=x_1$, so that beyond this plane $k$ retains the value it has at that point, $k\,|^{x\geq x_1} = k_1$, say, the solution for $x>x_1$ can be written

$$U = Ce^{-ik_1x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

there being no reflected wave in the homogeneous medium $x>x_1$.

* *Phys. Zeitschr.* xvii. pp. 369–373, (1916) ; xviii. pp. 35–41 (1917).
† Proc. Camb. Phil. Soc. xxii. pt. 2, pp. 124–131 (1924).

The boundary conditions of continuity require that

$$\frac{1}{U}\frac{dU}{dx}\bigg|^{-0} = \frac{1}{U}\frac{dU}{dx}\bigg|^{+0}, \quad \ldots \quad (7)$$

$$\frac{1}{U}\frac{dU}{dx}\bigg|^{x_1-0} = \frac{1}{U}\frac{dU}{dx}\bigg|^{x_1+0}; \quad \ldots \quad (8)$$

and, further,

$$\psi|^0 = 0, \quad \frac{d\psi}{dx}\bigg|^0 = k_0, \quad \frac{d\psi}{dx}\bigg|^{x_1} = k_1. \quad \ldots \quad (9)$$

Writing now

$$K/H=\beta, \quad A/B=\alpha, \quad \frac{d\phi}{dx}\bigg|^0 = f_0, \quad \frac{d\phi}{dx}\bigg|^{x_1} = f_1, \quad \psi\bigg|^{x_1} = \psi_1,$$

we have the conditions

$$-ik_0(1-\beta)/(1+\beta) = f_0 + ik_0(\alpha-1)/(\alpha+1), \quad \ldots \quad (10)$$

$$-ik_1 = f_1 + ik_1(\alpha e^{i\psi_1} - e^{-i\psi_1})/(\alpha e^{i\psi_1} + e^{-i\psi_1}), \quad (11)$$

and find, from (11),

$$\alpha = -f_1 e^{-2i\psi_1}/(f_1 + 2ik_1), \quad \ldots \quad (12)$$

which, on substituting in (10), gives

$$\beta = \frac{f_0 f_1(1-e^{-2i\psi_1}) + 2i(k_1 f_0 - k_0 f_1 e^{-2i\psi_1})}{-f_0 f_1(1-e^{-2i\psi_1}) - 2i(k_1 f_0 - k_0 f_1) - 4k_0 k_1} \quad \ldots \quad (13)$$

symbolically; the ratio of the intensities of the reflected and incident waves becomes, after reduction,

$$|\beta|^2 = R/(R + 4k_0^2 k_1^2), \quad \ldots \quad (14)$$

where

$$R = (k_1 f_0 - k_0 f_1)^2 + f_0 f_1 \{(2k_0 k_1 + \tfrac{1}{2} f_0 f_1)(1-\cos 2\psi_1) + (k_1 f_0 - k_0 f_1)\sin 2\psi_1\}. \quad (15)$$

The phase-difference is

$$\delta = \tan^{-1}\frac{2(k_1 f_0 - k_0 f_1 \cos 2\psi_1) + f_0 f_1 \sin 2\psi_1}{f_0 f_1(1-\cos 2\psi_1) - 2k_0 f_1 \sin 2\psi_1} - \tan^{-1}\frac{2(k_1 f_0 - k_0 f_1) + f_0 f_1 \sin 2\psi_1}{f_0 f_1(1-\cos 2\psi_1) + 4k_0 k_1}.$$

It is seen from the expressions (14) and (15) that the reflected intensity depends mainly on the values $f_0$ and $f_1$, so that, if $f_1=0$, there is no reflexion in the variable medium,

and the reflected wave in the medium $x<0$ depends entirely on the conditions at the boundary $x=0$ ; on the other hand, if $f_0=0$, the reflexion depends wholly on the boundary values $k_1$ and $f_1$ ; if both $f_0$ and $f_1$ are finite, the thickness of the layer comes in through the terms $1-\cos 2\psi_1$, $\sin 2\psi_1$, entailing a certain "fluctuation" of intensity, with minima for such values of $x$ as make $\psi_1=m\pi$, $m$ being any integer. It also appears, from the equation (4) and the condition (9), that $f_0$ and $f_1$ depend ultimately on the values of $dk/dx$ at the boundaries, and this is brought out more clearly by the following.

The equations (3) and (4) may obviously be satisfied by taking

$$\frac{d\psi}{dx}=k, \quad \psi=\int_0^x k\,dx, \quad \text{and } \phi=-\tfrac{1}{2}\log k, \quad (16)$$

provided

$$3\left(\frac{dk}{dx}\right)^2=2k\frac{d^2k}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which is true, exactly, if

$$k=k_0(1+ax)^{-2}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If $k$ is continuous, we may take the development

$$k=k_0+k'x+\ldots \quad . \quad . \quad . \quad . \quad . \quad (19)$$

as correct to the first order for sufficiently small $x$, and we may also choose "$a$" in (18) so that the development

$$k_0(1+ax)^{-2}=k_0-2k_0ax+\ldots \quad . \quad . \quad . \quad (20)$$

will fit (19) as closely as we please. Hence, for sufficiently small $x$, we have

$$\frac{d\psi}{dx}=k,$$

$$\therefore \quad f_0=-\frac{1}{2k_0}\frac{dk}{dx}\Big|^0, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and a similar argument applies, on account of symmetry, to the boundary $x=x_1$.

It may therefore be inferred that reflexion from a continuously varying transition-layer depends primarily on the discontinuity of the gradient of the refractive index ; if both the refractive index and its gradient are continuous throughout there is no reflexion. A similar conclusion was drawn

by Gans for the approximate solutions discussed (*l. c.*), and is not contradictory to Lord Rayleigh's result in so far as the form of variation considered (*l. c.*) does not admit of a vanishing gradient for any finite $x$.

The form referred to corresponds, with a slightly different notation, to

$$k = k_0(1+ax)^{-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

leading to

$$\phi = \tfrac{1}{2}\log(1+ax), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\psi = (k_0^2/a^2 - \tfrac{1}{4})^{1/2}\log(1+ax); \quad . \quad . \quad . \quad (24)$$

and thus,

$$f = \tfrac{1}{2}a(1+ax)^{-1}$$

$$= -\frac{1}{2k}\frac{dk}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If the function defined by (18) be adopted, we obtain, as the general solution,

$$U = k^{-1/2}\left(Ae^{i\int_0^x k\,dx} + Be^{-\int_0^x k\,dx}\right), \quad . \quad . \quad (26)$$

which may be used as approximate, in case $(dk/dx)^2$ and $k^{-1}\,d^2k/dx^2$ are small of the second order. This approximate solution has been discussed by Nicholson *. (*Cf.* also, Gans, *l. c.*)

Another case of interest is where

$$k = k_0 + k'x + \ldots$$

may be taken as correct to the first order, $k'$ being small. The solution may be approached by writing

$$\phi + i\psi = \Phi = \pm ik_0x + px + \tfrac{1}{2}rx^2 + \ldots, \quad . \quad . \quad (27)$$

where $p$ and $r$ are small constants of the first order. The function $e^{\Phi}$ will satisfy the equation (1), provided

$$\frac{d^2\Phi}{dx^2} + \left(\frac{d\Phi}{dx}\right)^2 + k^2 = 0, \quad . \quad . \quad . \quad . \quad (28)$$

and on substituting, and equating to zero coefficients to like

* Proc. Roy. Soc. A, lxxxi. pp. 286–299 (1908).

powers of $x$, the coefficients are found subject to the (approximate) conditions

$$\left.\begin{aligned} \pm ik_0 r + k_0 k' &= 0, \\ r \pm 2ik_0 p &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (29)$$

which are satisfied by taking

$$r = \pm ik' \qquad p = -k'/2k_0.$$

The solution may thus be written

$$U = e^{-k'x/2k_0}\{Ae^{i(k_0x+\frac{1}{2}k'x^2)} + Be^{-i(k_0x+\frac{1}{2}k'x^2)}\}. \quad . \quad (30)$$

We now have $f_0 = f_1 = -k'/2k_0$, and so the intensity ratio becomes

$$|\beta|^2 = \frac{k'^2\{(k_1-k_0)^2 + 2k_0k_1(1-\cos 2\psi_1)\}}{k'^2\{(k_1-k_0)^2 + 2k_0k_1(1-\cos 2\psi_1)\} + 16k_0^4k_1^2}, \quad (31)$$

neglecting, in this expression, cubes and higher powers of $k'$.

Introducing the refractive indices and taking $\mu_0 = 1$, we find

$$|\beta|^2 = \frac{\mu'^2(\mu^2 - 2\mu\cos 2\psi_1 + 1)}{\mu'^2(\mu^2 - 2\mu\cos 2\psi_1 + 1) + 64\pi^2n^2\mu^2/c^2}. \quad . \quad (32)$$

The reflected intensity is thus always small, as was to be expected from the assumed smallness of $k'$, and it nearly vanishes for values of $x$ such that

$$\psi = k_0x + \tfrac{1}{2}k'x^2 = m\pi,$$

or, in terms of the wave-length $\lambda$,

$$(2\pi/\lambda_0)(x - \lambda'x^2/2\lambda_0) = m\pi,$$

leading to

$$x = (\lambda_0/\lambda')(1 \pm \sqrt{1 - 4m\lambda'}) \quad . \quad . \quad . \quad . \quad (33)$$

Even a small change in the thickness of the layer may thus cause an appreciable relative change in the reflected intensity.

XI. *Note on the Application of the Ultramicrometer to the Microbalance.* *By* R. Whiddington, *M.A., D.Sc., Cavendish Professor of Physics, and* F. A. Long, *B.Sc., Assistant Lecturer in the University of Leeds* *.

UP to the present the most sensitive microbalances which have been constructed are those of Steele and Grant †, Ramsay and Gray ‡, and Pettersson §. In all cases, however, great sensitiveness has been obtained only at the expense of the maximum load and the general robustness of the apparatus, and the smallest change in weight measurable was of the order of one ten-millionth of the total load.

They have certain points in common. In each case the balance proper was made entirely of fused quartz, and enclosed in an airtight case, so that small changes in weight might either be compensated by or compared with the effective change in weight of a small air-bulb, produced by altering the pressure in the case. Deflexions of the beam were observed by the reflexion of a beam of light from a small mirror on the balance.

The first two, however, were supported by knife-edges, while the last was suspended by two very fine quartz fibres.

With the knife-edge type of balance the sensitiveness is no doubt limited by the sharpness of the knife-edge, but with the quartz suspensions, which work with no friction, the limit, as Pettersson suggests, appears to be imposed solely by the smallness of the tilt which can be measured by the optical device employed. Thus finally the problem resolves itself into one of measuring an extremely small distance, since the measurement of the distance moved through by the end of the beam amounts to the same thing as measuring the angle of tilt.

Now, the "Ultramicrometer" ‖ devised by one of us provides a means of measuring distances of an extremely minute order.

This note is an account of an attempt to apply the "Ultramicrometer" to a balance of the Pettersson type.

Two possibilities were borne in mind: one being the construction of a balance of greater absolute sensitiveness than those mentioned, and the other being the construction of one

* Communicated by the Authors.
† Proc. Roy. Soc. A. lxxxii. p. 580.
‡ Proc. Roy. Soc. A. lxxxiv. p. 536.
§ Proc. Phys. Soc. xxxii. p. 209 (1920).
‖ Phil. Mag. xl. p. 634 (1920).

of greater relative sensitiveness. Our experimental arrangements made it more convenient to attack the second problem first.

Suppose that we have a balance with a beam about 9 cm. long, one end of which descends through a distance of the order of $10^{-3}$ cm. for an increase in weight of $10^{-6}$ gm. It was estimated that an ultramicrometer attachment, such as could be adapted to this balance, would measure a change of $2 \times 10^{-7}$ cm., so that one might reasonably expect by this method to measure changes in weight of the order of $10^{-9}$ gm. with a much heavier balance than those mentioned, and hence a greater load.

### *Principle of the Instrument.*

The "Ultramicrometer" has been fully described previously in this Magazine *, but it may be mentioned that it consists essentially of two similar thermionic valve circuits capable of maintaining continuous oscillations of a frequency of some millions. The frequencies may be made to differ by such an amount that the beats produced by the two together give a note of audible frequency in a telephone receiver. Any slight change in capacity in one circuit produces a change in its frequency and hence a change in the frequency of the audible note.

In principle the method is merely to attach to each end of the balance light metal plates opposite and parallel to two fixed ones, thus in effect producing two condensers, one at each end of the beam.

A slight tilt of the beam has the effect of increasing one capacity while decreasing the other, the frequency of one circuit is thereby decreased and that of the other increased, so that the audible note changes. Incidentally, the total change is double what it would be if the movement were only applied to a condenser in one circuit.

Actually, of course, this arrangement involves two electrical connexions to the two plates on the beam. This is undesirable and is avoided in the final form of the instrument by a device described later.

### *General description of the Balance.*

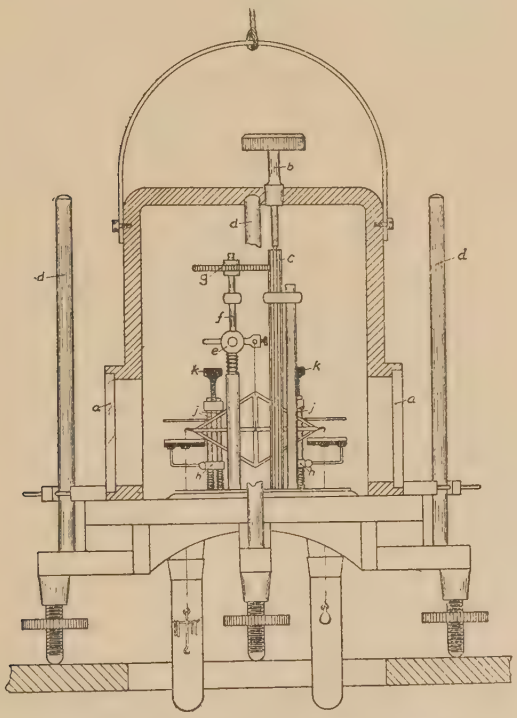
Fig. 1 is a diagrammatic sketch of the balance as actually used, with the case shown in section.

The balance is enclosed in a thick brass cylindrical case,

* *Loc. cit.*

provided with two windows (*a*). This case is flanged and ground on to a brass base, an airtight fit being obtained with the aid of a little rubber grease. Being too heavy to lift conveniently it is suspended by a wire passing over pulleys and its movement is controlled by the guide-rods (*d*).

Fig. 1.



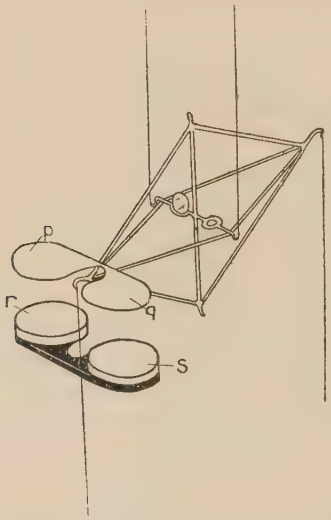

The beam, described more in detail later, is suspended from a horizontal rod capable of adjustment in two directions at right angles in a horizontal plane. The block (*e*) supporting this rod is made to move up and down the pillar (*f*) by the gear wheels (*g*), which are actuated from outside the case by the rod (*b*). The latter makes an airtight fit by means of a ground joint and engages in a slot in the rod (*c*).

*Particular details embodied in the design.*

(a) *The case.*—This is made of thick brass in order to minimize the effects of temperature changes.

(b) *The beam.*—On account of its physical properties fused quartz was used for the beam, as in other microbalances. The beam is shown separately in fig. 2. The central transverse rod has a small mirror cemented on one side, to permit of the use of a lamp and scale in the adjustment, and a tiny nut working on a fine screw cemented on the other side. This nut, which has a small projecting arm to enable it to be turned easily, is for rapidly making fine adjustments of the centre of gravity.

Fig. 2.




(c) *The suspension.*—The beam is suspended by two very fine quartz fibres, drawn out of the ends of the central transverse rod itself, and these are cemented to the support mentioned in the general description.

(d) *The air-weight and counterpoise.*—From each end of the beam a fibre ending in a hook is drawn. One carries a small air-bulb of about ·3 c.c. external volume, and the other a quartz counterpoise made in a form suitable for attaching additional quartz weights. These hang in closed glass tubes fitted by ground joints to the base. To make use of the air-weight the case of course has to be evacuated, so a side tube is provided and connected with an exhaust-pump and mercury manometer.

(e) *Condensers attached to the beam.*—To avoid any direct electrical connexion to the beam, the single condenser at

each end, mentioned earlier, was replaced in the instrument as used by two condensers in series. Fig. 2 shows one pair of condensers only. The two middle plates ($p$ and $q$) of each pair are cut from a single piece of aluminium sheet, as thin as possible consistent with rigidity, and cemented by the connecting portion to the beam. The two extreme plates ($r$ and $s$) of each pair form the fixed plates, which are arranged under the connected plates and mounted on ebonite blocks. These blocks are connected by stiff ball and socket joints to metal blocks ($h$) moving on parallel pillars ($j$), and are moved by the screws ($k$). From the fixed plates leads pass through insulating plugs in the base to the inductances of the oscillating circuits.

The fixed plates also serve as an arrestment for the beam.

### *The Electrical Circuit.*

The electrical circuit is essentially the same as that described in the account of the "Ultramicrometer"*, the condensers being incorporated with the balance, as explained above. Instead of putting the telephone-receiver directly in the plate circuits of the valves, a three-valve low-frequency amplifier and a loud-speaker were used. In order to bring the beat-note to a suitable audible frequency, a parallel plate air-condenser, one plate of which can be moved by a micrometer screw, was connected in parallel with the capacity in one of the valve circuits. Additional control can be obtained if necessary by varying the pressure in the case.

### *Setting up the Balance.*

Setting up and adjusting the balance are operations requiring some patience, chiefly owing to the accidental breakage of one or other of the quartz fibres, and the fact that condenser plates have to be cemented to the beam.

### *Precautions necessary with the Electrical Circuits.*

The extremely sensitive nature of the "Ultramicrometer" gives rise to certain difficulties which are briefly enumerated below, with the steps taken to minimize them.

($a$) Mechanical vibration was prevented as far as possible by placing the apparatus on thick rubber corks on a rigid slate bench in a room in the basement. Even so, trouble

* *Loc. cit.*

was traced to movements and disturbances in other parts of the building and, in fact, outside it, so that it was found necessary to work very late at night or in the early morning.

(*b*) All leads were of stiff enamelled copper wire to prevent vibration.

(*c*) By suitably spacing the leads and portions of the apparatus stray capacity effects were minimized, as they might become appreciably large in the aggregate compared with the condensers used, thereby decreasing the sensitiveness.

(*d*) The apparatus was effectively screened against capacity changes due to movements of the observer by earthing the case and the negative side of the L.T. battery, and by enclosing other portions in earthed tin-lined cases.

(*e*) The batteries should be in excellent condition. A slight change in filament current alters the frequency *, but it is found that the current can be adjusted so that either an increase or a decrease produces a change of note in the same direction, *i. e.* this occurs at a turning point on a frequency filament-current curve (with anode voltage constant). The curve is fairly flat at this turning point, so that at this critical adjustment any slight fluctuation due to the batteries is less likely to affect the note.

There is a similar point on the frequency-anode voltage curve (with constant filament current) †, but in this particular instance it was not possible to take advantage of this fact owing to the necessity for keeping the anode voltage as low as possible, in order to minimize the electrostatic attraction between the plates.

(*f*) Soldered joints were used throughout.

(*g*) The cellar room temperature was found to be sufficiently constant for the experiment.

(*h*) Owing to the inequality in the separation of the condenser plates, there was an appreciable deflexion due to electrostatic attraction when the valves were switched on. This was made as small as possible by keeping the anode voltage low and by altering the pressure slightly until the plates were more equally spaced.

This additional force which is introduced into the balance system proved to be greater than expected, but as the plates can be almost equally spaced, and as the changes in weight it was desired to measure produce only a minute tilt, it did not interfere with the working of the balance.

* Proc. Roy. Soc. A. xcvi. p. 455, and A. xcvii. p. 191.

† In some unpublished work carried out in this laboratory by Miss C. E. Watson, M.Sc., it was shown that the frequency falls to a minimum value and then rises again when the anode voltage is varied.

*Method of using the Balance.*

The centre of gravity was adjusted until the balance had a sensitiveness of the order mentioned earlier.

On varying the pressure the deflexion of the light-spot was found to be reasonably uniform, with the zero constant over a considerable period, showing however a slight creep in several days.

The constancy of zero was further tested by altering the anode voltage and bringing it back to its original value several times.

A long piece of very fine constantan wire was weighed and measured carefully, and a very small portion was cut off and measured. The change in pressure necessary to counterbalance this was used as the standard. It agreed sufficiently well with the value calculated from the volumes of the bulb and counterpoise.

A vessel of known capacity (about 1 litre) was also connected to the apparatus through a tap, and by measuring the change in pressure when this vessel was connected with the rest of the apparatus at various pressures, the internal volume of the case etc. could be calculated.

As a change of 1 mm. pressure produced a greater change in weight than was required, an additional adjustable manometer was attached and the bore of the fixed limb found. A small movement of the open limb produced a slight known change in volume from which the change in pressure could be calculated since the volume was known. The change in weight corresponding to this is known, so that we were able to produce changes in weight of the order desired. Of course the same end could have been attained by using a smaller bulb, but it was found more convenient to retain the larger bulb and use the method described.

With the valve circuits in action the note was adjusted to such a frequency that it gave a few beats per second with the note emitted by an audible frequency circuit used as standard, also provided with a loud-speaker so that the beats could be heard some distance away.

The pressure was then varied by steps over a small range, and the standard circuit adjusted each time by altering its capacity.

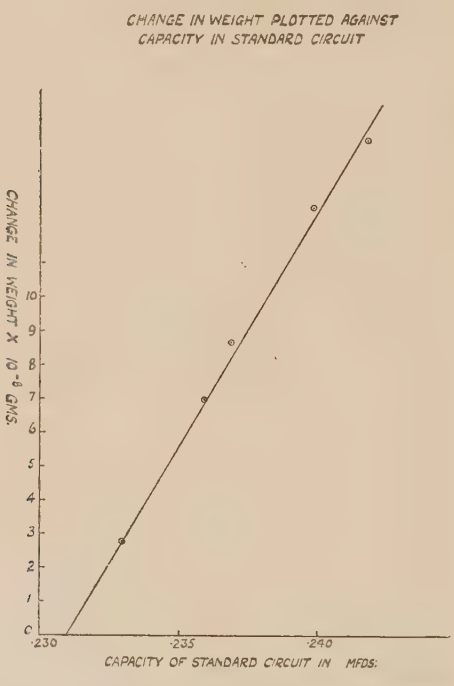
*Results obtained.*

As the note varies somewhat until the circuit has been left in operation for some minutes, time must be allowed for heat effects to become steady.

The slow beats were found to vary periodically about some mean, presumably owing to the natural swing of the balance which was unexpectedly persistent, and it was consequently impossible to count them. On this account the fullest use could not be made of the "Ultramicrometer" principle in the application of which the slow beat method is essential for highest accuracy.

The standard note was therefore adjusted to give the mean note each time the pressure was varied, and fig. 3 shows the

Fig. 3.




kind of results obtained, where the change in weight is shown plotted against the corresponding capacity in the standard circuit. The points lie fairly well on a straight line, which is to be expected if the movement of the condenser plates is small compared with their separation.

The slopes of these lines are all practically the same, and the same note could be restored by returning the pressure to its original value.

The smallest change in weight which was used in obtaining the curves was $1{\cdot}67 \times 10^{-8}$ gm., which is rather less than

one ten-millionth of the load, that is to say this arrangement is a few hundred times more sensitive than with a lamp and scale.

A change in weight of about $5 \times 10^{-9}$ gm. produced an unmistakable change in the number each time, but since this change was of the same order as the variation of the note about the mean, it was impossible to *measure* it with any accuracy.

### *Conclusion.*

Our experience indicates that if the slow beat method could have been used it would have been possible to measure a change of as little as $10^{-9}$ gm. or even less in a load of about 0·2 gm.

Unfortunately the practical difficulties have forced us to the conclusion that the instrument is not sufficiently simple in operation to be generally useful. Apart from this the balance is not sufficiently sensitive for the purpose for which it was originally designed.

We wish to thank Messrs. Ludbrook and Beecroft for their assistance in the workshop and Mr. L. G. Stanton for preparing the diagrams. Part of the cost of the balance was defrayed by a grant from the Royal Society.

Physics Laboratories,
University of Leeds,
September 1924.

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## XII. *On the Elastic Impact of the Pianoforte Hammer.*—II.

*By* S. Bhargava *and* R. N. Ghosh, *Department of Physics, Allahabad University* *.

In the June number of the 'Philosophical Magazine,' vol. xlvii., the authors have shown that the elasticity of the hammer felt must be taken into consideration for calculating the duration of impact. Kaufmann's approximate theory with the additional consideration of the elasticity of the hammer felt agrees fairly with the experimental results. In the present paper, calculation is made of the transmission of energy to the string by the elastic hammer, and the effect of stiffness upon the duration of impact has been shown to diminish it, whatever be the tension of the string. Finally, the law of pressure and the resulting amplitude of the

* Communicated by Prof. M. N. Saha.

partials have been calculated. It is found that Helmholtz's term plus two other terms are present, which are not insignificant.

### 2. *Calculation of the Communication of the Energy to the String.*

*Helmholtz's Case.*

In this case it is supposed that the impact is over before the disturbance produced by the hammer has travelled outwards. The pressure $p$ has been shown to be given by

$$p = v_0(\mu M)^{\frac{1}{2}} \sin \{\mu/M\}^{\frac{1}{2}}t, \quad . \quad . \quad . \quad . \quad (1)$$

where $v_0$ is the initial velocity of the hammer. The duration of the impact is given by $\pi(\mu/M)^{-\frac{1}{2}}$. The displacement of the hammer is given by

$$y_0 = v_0(M/\mu)^{\frac{1}{2}} \sin \{\mu/M)^{\frac{1}{2}}t\}. \quad . \quad . \quad . \quad . \quad (2)$$

The velocity of the hammer when it leaves the string is easily found to be the same as the initial velocity:

$$\text{Velocity of rebound} = -v_0.$$

Thus we see that in the ideal case of localized disturbance there should be no loss of energy by the hammer; but that energy is lost in producing waves in the piano string is beyond doubt: hence the assumption of no wave-motion during impact requires modification. The expressions (1), (2), and (3) are different from those derived by Mr. W. H. George*, who by introducing a new quantity, namely the velocity of rebound $v$, has obtained at $t=\frac{T}{2}$ the value of maximum pressure,

$$p = \frac{M\pi}{2T}(v_0+v),$$

where T is the duration of impact, while from (1) we obtain

$$p = \frac{v_0 M\pi}{T}.$$

Further, at $t=\frac{T}{2}$ velocity of the hammer should be zero *when the pressure is maximum*. These facts are represented by (1) and (2), while George's expression gives for the velocity of the hammer at the same instant $\frac{1}{2}(v_0-v)$. If,

* Phil. Mag. xlvii. p. 596 (1924).

however, the above condition be imposed on Helmholtz's expression for the pressure,

$$p = \mathrm{F} \sin \pi t/\mathrm{T},$$

$$\mathrm{F} = p_0 = \frac{v_0 \mathrm{M} \pi}{\mathrm{T}},$$

then George's equation (2) will be modified as follows :

$$\frac{dy}{dt} = v_0 + v_0(\cos \pi t/\mathrm{T} - 1),$$

from which it is evident that at $t = \mathrm{T}$ velocity of rebound is $-v_0$. It is the same as the initial velocity since the loss of energy of the hammer $= \frac{1}{2}\mathrm{M}v_1^2 - \frac{1}{2}\mathrm{M}v_2^2$, where $v_1$ and $v_2$ are velocities before and after impact. In this case $v_1 = v_2$, and therefore there is no loss.

In a case when the hammer spring is supposed to be bounded by a cylindrical surface which impinges upon the string, producing a localized dimple of the same size, we have

$$p = \mathrm{T}(8y/a)^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $a$ is the radius of the impinging cylinder. The equation of motion of the hammer is given by

$$\mathrm{M} \frac{d^2 y_0}{dt^2} = -p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $p$ is the pressure between the hammer and the spring. Further,

$$p = \mu \xi \quad \text{and} \quad y_0 = y + \xi. \quad . \quad . \quad . \quad . \quad (5\,a)$$

$y$ and $y_0$ are displacements of the string and hammer ; $\xi$ is the compression and $\mu$ the strength of the spring. From (4), (5), and (5 $a$) we obtain to a first approximation,

$$y_0 = y + \alpha \xi^2,$$

where

$$\alpha = \frac{a\mu^2}{8\mathrm{T}^2} \quad \text{and} \quad (1 + 2\alpha\xi) \frac{d^2\xi}{dt^2} + 2\alpha \left(\frac{d\xi}{dt}\right)^2 + n^2 \xi = 0,$$

where $n^2 = \mu/\mathrm{M}$.

For small values of $\alpha$, an approximate solution for $\xi$ is

$$\xi = v_0/n \sin nt - \frac{2\alpha v_0^2}{n^2} \cos nt + \frac{2\alpha v_0^2}{n^2}.$$

$v_0$ the initial velocity is supposed to be small compared with $n$.

The duration of impact is measured by the interval from the beginning of impact to the instant when the pressure is again zero. In this case it is $T=(\pi/n+4\alpha v_0/n^2)$ and

$$\dot{y}=\frac{d\xi}{dt}+2\alpha\xi\frac{d\xi}{dt}.$$

At $t=0$, $\dot{y}=v_0$ and $y=0$;

at $t=T/2$, $\dot{y}=0$ and $y=\frac{v_0}{n}\left(1+\frac{2\alpha v_0}{n}-\frac{16v_0\alpha^4}{n^4}+\ldots\ldots\right)$;

at $t=T$, $\dot{y}=-v_0$ and $y=0$.

Hence, when the hammer rebounds, no wrinkle is left in the string, and in this case, too, there is no loss of energy by the hammer. The case is different, however, when the wrinkle does not remain localized during impact.

*Kauffmann's Case.*

Kauffmann considered a case when the length of the string was finite and it was displaced during impact. He took the case when the striking distance $\alpha$ was small, but he regarded the hammer as hard. We have shown that the elasticity of the hammer felt must be taken into consideration. Regarding the hammer felt as equivalent to an elastic spring of strength $\mu$, and neglecting terms of the order $10^{-3}$ in comparison with others, we obtain a solution for the case of the elastic hammer:

$$y=Ae^{-\frac{K}{2}t}\sin qt, \qquad \ldots\ldots\ldots \quad (6)$$

$$q^2=\left[\frac{T}{\alpha M_1}\left(\frac{\mu\alpha}{T+\mu\alpha}\right)-\tfrac{1}{4}K^2\right],$$

$$K=\frac{T}{M_1C}\left(\frac{\mu\alpha}{T+\mu\alpha}\right), \qquad \ldots\ldots\ldots \quad (7)$$

and

$$A=\frac{v_0}{q}\left\{1+\frac{T}{\mu}\left(\frac{1}{\alpha}-\frac{K}{2C}\right)\right\}^{-1}.$$

The duration of impact is approximately given by

$$\frac{\pi}{q}+\delta t,$$

where $\delta t=\alpha/C$ approximately.

Further,

$$y_0 = \mathrm{A}\left(1+\frac{\mathrm{T}}{\mu\alpha}\right)e^{-\frac{\mathrm{K}t}{2}}\sin qt \quad . \quad . \quad . \quad . \quad (8)$$

and $$\frac{dy_0}{dt} = \mathrm{B}e^{-\frac{\mathrm{K}t}{2}}\left\{q\cos qt - \frac{\mathrm{K}}{2}\sin qt\right\},$$

where $$\mathrm{B} = \mathrm{A}(1+\mathrm{T}/\mu\alpha), \quad v_0 = \mathrm{B}q,$$

and $v$, the velocity of rebound,

$$= \mathrm{B}qe^{-\frac{\pi\mathrm{K}}{2q}} \text{ approx.}$$

The energy absorbed by the spring

$$= \tfrac{1}{2}\mathrm{M}_1 v_0^2 - \tfrac{1}{2}\mathrm{M}_1 v^2$$

$$= \tfrac{1}{2}\mathrm{M}_1\mathrm{A}^2q^2(1-e^{-\frac{\pi\mathrm{K}}{q}}) \text{ approx.}$$

The loss of energy in producing waves can be calculated by evaluating the potential energy of the wave-motion in the string. The equation of wave-motion is

$$\frac{d^2y}{dt^2} = \mathrm{C}^2\frac{d^2y}{dx^2}.$$

Neglecting the effect of the reflexions from the shorter end, we must have

$$\text{at } x=\alpha, \quad y=\mathrm{A}e^{-\frac{\mathrm{K}t}{2}}\sin qt,$$

$$\text{and at } x=\infty, \quad y=0.$$

Assuming $y \propto e^{i(qt+\mathrm{K}it/2)}$,

$$\frac{d^2y}{dx^2} = -(q+i\mathrm{K}/2)^2/\mathrm{C}^2.y\ ;$$

$$\therefore \quad y = \mathrm{A}e^{-\frac{\mathrm{K}}{2\mathrm{C}}(\mathrm{C}t-x+\alpha)}\sin\frac{q}{\mathrm{C}}(\mathrm{C}t-x+\alpha). \quad . \quad (9)$$

This holds true only up to $0<t<l-\alpha/\mathrm{C}$.

The total energy of wave-motion (kinetic plus potential)

$$= \mathrm{T}\int_0^{\mathrm{C}T}\left(\frac{dy}{dx}\right)^2 dx = \frac{\mu_1\mathrm{A}^2q^2}{2}\left(1-e^{-\frac{\pi\mathrm{K}}{q}}\right) \text{ approx.}$$

It is found that the energy is the same as lost by the hammer after impact.

### 3. *Effect of Stiffness on the Duration of Impact.*

The effect of stiffness on the free vibrations of a finite string has been investigated by the late Lord Rayleigh *.

* 'Theory of Sound,' vol. i. art. 137.

He found out the increase in the potential energy due to stiffness, and then calculated the effect upon the periods. The same can, however, be done by an alternative method, which will be employed to evaluate the effect upon the duration of impact.

$$M = EI\frac{d^2y}{dx^2}, \quad \frac{dM}{dx} + F = 0.$$

M = bending moment, E = Young's modulus, I = moment of inertia of the cross-section perpendicular to the plane of bending, F = the shearing force.

The equation of motion of a stretched string now takes the form

$$\rho\frac{d^2y}{dt^2} = T\frac{d^2y}{dx^2} + \frac{\delta F}{\delta x}.$$

Assume

$$y \propto \sin\frac{s\pi x}{l}.\phi,$$

where $\phi$ is the normal coordinate.

$$\ddot{\phi} = -\frac{T}{\rho}\frac{s^2\pi^2}{l^2}\left\{1 + \frac{EI}{T}\frac{s^2\pi^2}{l^2}\right\}\phi,$$

$$T_s' = T_s\left\{1 - \frac{EI}{2T}\frac{s^2\pi^2}{l^2}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where $T_s'$ is the period when stiffness is considered, and $T_s$ is the period for a perfectly flexible string. Equation (10) is the same as formula (2) of art. 137 of Rayleigh's 'Theory of Sound,' vol. i.

Returning to our problem, we observe that

$$\left.\begin{aligned} y_1 &= \frac{yx}{\alpha} \qquad 0 < x < \alpha, \\ y_2 &= f(Ct - x) \quad \alpha < x < l, \\ y &= f(Ct) \qquad x = \alpha, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10\,a)$$

and

$$p = \mu\xi = T\left\{\frac{y}{\alpha} + \frac{1}{C}\frac{dy}{dt}\right\} + F_1,$$

where $F_1$ is the shearing force due to stiffness on account of the change of inclination from $y_1$ to $y_2$ in the neighbourhood of the point struck.

$$M = EI\left\{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2\right\},$$

$$F = -\frac{dM}{dx} = EI\frac{d^2y}{dx^2} = \frac{EI}{C^2}\frac{d^2y}{dt^2}, \quad . \quad . \quad . \quad (11)$$

where $\left(\frac{dy}{dx}\right)_1$ and $\left(\frac{dy}{dx}\right)_2$ are the inclinations on the two sides.

The equation (12) * is modified to the form

$$M_1\left(1+\frac{T}{\mu\alpha}-\frac{EI}{C^2M}\right)\frac{d^2y}{dt^2}+\frac{T}{C}\frac{dy}{dt}+\frac{Ty}{\alpha} = -\frac{MT}{\mu C}\frac{d^3y}{dt^3}-\frac{EIM}{e^2\mu}\frac{d^4y}{dt^4}. \quad . \quad (12)$$

A full solution of (12) will be cumbrous, and so of little value. We shall be content with an approximate solution by neglecting terms which are comparatively small. The right-hand side can then be assumed to be zero. Then the solution is of the same form as before (equation 12, *loc. cit.*), but the numerical magnitude of $q_s$ now comes out :

$$q_s^2-q^2=\frac{TEI}{C^2\alpha M^2}\left(T+\frac{\mu\alpha}{\mu\alpha}\right). \quad . \quad . \quad . \quad . \quad (13)$$

Hence we find that the direction of impact is decreased and that the diminution increases with I. For circular wire $I=\frac{\pi}{8}a^4$, so that the duration of the contact diminishes as the fourth power of the radius. Formula (13) shows that the duration of impact is always shortened, whatever be the nature of the hammer. Further, the effect is independent of tension.

### 4. *Calculation of the Law of Pressure.*

From (12) and (10 *a*) $p$ is easily found to be

$$p=ATe^{-\frac{K}{2}t}\left\{\frac{\sin qt}{\alpha}+\frac{R}{C}\cos\overline{qt+\phi}\right\}, \quad . \quad . \quad (14)$$

where

$$R=(q^2-K^2/4)^{\frac{1}{2}} \quad \text{and} \quad \tan\phi=K/2q.$$

From (14) it is observed that the pressure on the string is not zero at the instant of impact. It is equal to $TAq/C$, and at the end of the interval $\pi/q$ its magnitude is again the same, and becomes zero after a short interval $\delta t$ obtained by putting $t=\pi/q+\delta t$ in (14). The fact that the pressure is not zero at the instant of impact is also confirmed by Professor Raman's curves †, where the hammer is considered to be perfectly hard. The interval $\delta t$ is very small ; it is approximately equal to $\alpha/C$. Hence we shall neglect it, and consider the duration of impact to be given by $\pi/q$.

* Phil. Mag. xlvii. p. 1145 (June 1924).

† Raman & Banerjee, Proc. Roy. Soc. xcvii. A, p. 105.

This amounts to neglecting the second term of equation (14); then the pressure is given by

$$p = \frac{AT}{\alpha} e^{-\frac{K}{2}t} \sin qt. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

*5. Calculation of the Intensity of the Partials.*

The equation of motion of a point on the string in terms of normal coordinates is given by

$$\ddot{\phi}_s + \left(\frac{s\pi C}{l}\right)^2 \phi_s = \frac{2}{l\rho} \Phi_s,$$

where $\Phi_s$ is the generalized component of the force. And

$$\Phi_s = A \sin \frac{s\pi\alpha}{l} e^{-\frac{K}{2}t'} \sin qt'.$$

If $\phi_s$ and $\ddot{\phi}_s$ both be zero initially, then for $t > \frac{\pi}{q}$,

$$\phi_s = \frac{2A \sin \frac{s\pi\alpha}{l}}{ln\rho} \int_0^{\frac{\pi}{q}} e^{-\frac{K}{2}t'} \sin n(t-t') \sin qt' \, dt'$$

$$= \frac{2A \sin \frac{s\pi\alpha}{l}}{ln\rho} \left[ \frac{2q \cos \frac{n\pi}{2q} \cdot \sin n\left(t - \frac{\pi}{2q}\right)}{q^2 - n^2} - \frac{K \sin \frac{n\pi}{2q} \cdot \cos n\left(t + \frac{\pi}{2q}\right)}{2(q-n)^2} \right.$$

$$\left. + \frac{K}{4} \frac{\pi}{q} \frac{\sin n\left(t + \frac{\pi}{q}\right)}{(q-n)} - \frac{K\pi}{4q} \frac{\sin n\left(t - \frac{\pi}{q}\right)}{(q+n)} + \ldots \right] \text{approx.}$$

[Notations follow Rayleigh's 'Theory of Sound,' vol. i. art. 134.]

The final solution for the displacement $y$ becomes, when we substitute the value of $n$ (to the same order of approximation),

$$y = \frac{2A l^2}{\rho\pi C} \sum_1^\infty \frac{\sin \frac{s\pi\alpha}{l} \cdot \sin \frac{s\pi x}{l}}{s(lq - s\pi C)} \left\{ \frac{2q \cos\left(\frac{s\pi^2 C}{2lq}\right) \sin\left[\frac{s\pi C}{l}\left(t - \frac{\pi}{2q}\right)\right]}{(lq + s\pi C)} \right.$$

$$- \frac{K}{2} \frac{\sin\left(\frac{s\pi^2 C}{2lq}\right) \cdot \cos\left[\frac{s\pi C}{l}\left(t - \frac{\pi}{2q}\right)\right]}{(lq - s\pi C)} + \frac{K}{4ql} \sin\left[\frac{s\pi C}{l}\left(t + \frac{\pi}{q}\right)\right]$$

$$\left. - \frac{K\pi}{4ql} \frac{\sin\left[\frac{s\pi C}{l}\left(t - \frac{\pi}{q}\right)\right]}{(lq + s\pi C)/(lq - s\pi C)} + \ldots \right\} . \quad . \quad . \quad . \quad (16)$$

Equation (16) shows the first term obtained by Helmholtz and other terms which appear on account of finite displacement of the string during impact, but on account of the presence K of their magnitudes are small. The convergence of the first two terms is of the order $s^{-3}$, but the last two terms converge more slowly. They increase if K increases—that is, when the striking distance $\alpha$ increases.

For a given striking distance the first term disappears for those partials whose periods are 2/1, 2/3, 2/5, 2/7 of the duration of impact; but the other three terms do not disappear, hence the existence of a feeble intensity of them is explained.

When the point struck is very close to the first node of the $s$th component, then the $s$th component will be absent; but if at the same time the duration of impact $\frac{\pi}{q}=\frac{l}{Cs}$ approx. then $\sin\frac{s\pi^2C}{2lq}=1$ and $\sin\frac{s\pi\alpha}{l}$, $\cos\frac{s\pi^2C}{2lq}$ are both of the same order of smallness: the first term and the third term completely disappear, but the second term does not. Hence we find that a component having a node at the point struck is not entirely absent if the duration of impact is half the period of vibration of the component. Hipkins and others have found this experimentally to be true.

A given component will converge more rapidly the greater the value of $q$, which decreases as $\alpha$ increases. According to Helmholtz, however, the convergence is independent of the striking distance.

## *Conclusions.*

These formulæ have been proposed tentatively, and the authors hope to verify them experimentally on a future occasion.

Physics Department,
Allahabad University,
Allahabad, India,
July 15th, 1924.

XIII. *Note on the Problem of the Quantitative Formulation of Bohr's Correspondence Principle.* *By* Richard C. Tolman, *Ph.D., Professor of Physical Chemistry and Mathematical Physics at the California Institute of Technology* *.

## I. Introduction.

It has been shown by Kramers † that the frequency of the light emitted when an atom passes from one quantum state to another may be regarded as an average of the frequencies which would be emitted, on the basis of the classical theory, by the various mechanically possible states intermediate between the initial and final quantum states. Bohr's principle of correspondence indicates that the intensity of the light emitted should also be calculable by a suitable average of the intensities which would be emitted by these intermediate states on the basis of the classical theory. In two recent articles, Hoyt ‡ has used six possible methods of obtaining this average, including one suggested by Kramers, to calculate the relative probabilities of transitions between various pairs of quantized Kepler orbits, and has tried to test his results by data on the intensity of X-ray and Balmer series lines. Unfortunately, the experimental information available was not sufficient to permit a definite decision as to the proper method of averaging

It is the purpose of this note to present the results obtained by using these different methods of averaging, to calculate the probability of transitions in the case of three very simple quantum systems with only a single degree of freedom—namely the harmonic oscillator, the rotating dipole, and the simplified hydrogen atom with circular orbits. These systems are so simple that the calculations can be made without approximations, and the significance of the results can perhaps be more readily appreciated than in more complicated cases. It is a matter of interest that only one of the methods of averaging leads to the value zero for the probability that an electron will fall into the nucleus from the innermost orbit in the hydrogen atom, the other methods leading to an infinite value for this probability.

* Communicated by the Author.

† Kramers, *Kgl. Danske Vidensk. Selsk. Skifter,* 8. Raekke, iii. 3, p. 327.

‡ Hoyt, Phil. Mag. xlvi. p. 135 (1923); *ibid.* xlvii. p. 826 (1924).

## II. Theory of the Calculation.

Consider the transition of a system of *one degree of freedom* between two stationary quantum states S′ and S″, which have for the phase integral ($\oint pdq$) the values

$$I' = n'h \quad \text{and} \quad I'' = n''h, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $n'$ and $n''$ are the quantum numbers corresponding to the initial and final states, and $h$ is Planck's constant. Let the value of the phase integral for mechanically possible intermediate states be given with the help of the auxiliary variable $\lambda$ by the equation

$$I = I'' + \lambda(I' - I''), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\lambda$ can assume values between unity and zero.

In accordance with the equation of Kramers, the frequency $\nu$ of the light emitted by the transition is connected with the frequency $\tau\omega$ of the corresponding harmonic in the Fourier expansion for the motion of the atom by the expression

$$\nu = \int_{\lambda=0}^{\lambda=1} (\tau\omega)\, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the corresponding harmonic is the one in which the order number $\tau$ is related to the quantum numbers $n'$ and $n''$ by the equation

$$\tau = n' - n'', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the fundamental frequency $\omega$ in the motion of the atom is related to energy E and phase integral I by the classical equation

$$\omega = \frac{dE}{dI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

On the basis of the classical theory, the average rate of energy emission corresponding to the $\tau$th harmonic and a particular direction of polarization would be

$$\left(\frac{dR}{dt}\right)_{Cl.} = g C_\tau^2 \tau^4 \omega^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the proportionality constant $g$ has the value $16\pi^4/3c^3$, and the quantity $C_\tau$ is the amplitude of the $\tau'$th harmonic in the Fourier analysis of the expression for the electric moment of the atom in the direction of polarization under consideration. On the basis of the quantum theory, the average rate of energy emission corresponding to a particular transition would be

$$\left(\frac{dR}{dt}\right)_{Qu.} = A_{n'}^{n''} h\nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $A_i''$ is the probability, as introduced by Einstein, that the atom will make the transition in question in unit time. In accordance with the correspondence principle, we should expect the quantum theory expression to be some form of average of the classical expression which might be expressed by the equation

$$A_i'' h\nu = \sum_{x,\,y,\,z} [g C_\tau^2 \tau^4 \omega^4]_{\text{av.}}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where the summation sign indicates the necessity of taking into account the amplitude of oscillation parallel to each of the three axes. Or, if in accordance with the spirit of the correspondence principle * we take Q as the amplitude of the "virtual oscillators" of frequency $\nu$, which for the purposes at hand could be substituted for the atoms, we might write

$$A_i'' h\nu = \sum_{x,\,y,\,z} [g Q^2 \nu^4] = \sum_{x,\,y,\,z} [g C_\tau^2 \tau^4 \omega^4]_{\text{av.}} \quad . \quad . \quad . \quad (9)$$

The following six possible methods of performing the indicated average, and thus determining the amplitude Q of the "virtual oscillator," are the ones to be investigated :—

$$Q^2 = \int_0^1 C_\tau^2 \, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

$$Q = \int_0^1 C_\tau \, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

$$\log Q = \int_0^1 \log C_\tau \, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

$$Q^2 \nu^4 = \int_0^1 C_\tau^2 \tau^4 \omega^4 \, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$

$$Q \nu^2 = \int_0^1 C_\tau \tau^2 \omega^2 \, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (E)$$

$$\log (Q \nu^2) = \int_0^1 \log (C_\tau \tau^2 \omega^2) \, d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (F)$$

## III. Expressions used in the Calculations.

In order to perform the indicated integrations, we shall have to express amplitude $C_\tau$ and frequency of motion $\omega$ in terms of the auxiliary variable $\lambda$. This can be easily done, however, since the relation between the phase integral I and the variable $\lambda$ is given by equation (2), and $C_\tau$ and $\omega$ are completely determined when the value of I is known.

We give below, for the three systems considered, all that

* See Bohr, Kramers, and Slater, Phil. Mag. xlvii. p. 785 (1924).

remains of the Fourier expansions for the electric moments $\xi$, $\eta$, and $\zeta$ parallel to the three axes, the amplitude C and frequency of motion $\omega$ for mechanically possible motions lying between the states determined by the quantum numbers $n$ and $n+1$, and the frequency $\nu$ of the light emitted by a quantum transition from $n+1$ to $n$. For convenience, the results are expressed in terms of the quantities $C_1$ and $\omega_1$, which are the amplitude and frequency of motion in the quantum state where $n$ equals unity.

1. *The Linear Harmonic Oscillator.*

$$\xi = C\cos 2\pi\omega t, \qquad \eta = \zeta = 0;$$
$$C = (n+\lambda)^{1/2} C_1;$$
$$\omega = \omega_1;$$
$$\nu = \omega_1.$$

2. *The Rigid Dipole Rotator.*

$$\xi = C\cos 2\pi\omega t, \quad \eta = C\cos\left(2\pi\omega t+\frac{\pi}{2}\right), \quad \zeta = 0;$$
$$C = C_1;$$
$$\omega = (n+\lambda)\,\omega_1;$$
$$\nu = \frac{(2n+1)}{2}\,\omega_1.$$

3. *The Simplified Hydrogen Atom (Electron in Circular Orbit).*

$$\xi = C\cos 2\pi\omega t, \quad \eta = C\cos\left(2\pi\omega t+\frac{\pi}{2}\right), \quad \zeta = 0;$$
$$C = (n+\lambda)^2 C_1;$$
$$\omega = \frac{\omega_1}{(n+\lambda)^3};$$
$$\nu = \frac{2n+1}{n^2(n+1)^2}\,\frac{\omega_1}{2}.$$

RESULTS OF THE CALCULATIONS.

The results of the calculations for the three systems are presented in tabular form. The first column gives the method of averaging employed; the second column the square of the amplitude of the "virtual oscillator," $Q^2$, corresponding to the transition from quantum number $n+1$ to $n$; the third column the probability, $A^n_{n+1}$, that this transition will occur in unit time; and the last two columns the values assumed by these two quantities for the special case of the transition from quantum number one to zero.

1. *The Linear Harmonic Oscillator.*

| Method of Average. | $Q^2$ $n+1\to n$. | $A^n_{n+1}$. | $Q^2$ $1\to 0$. | $A_1{}^0$. |
|---|---|---|---|---|
| A and D. | $\frac{2n+1}{2} C_1^2$ | $\frac{2n+1}{2} \frac{g}{h} C_1^2\omega_1^3$ | $\frac{1}{2} C_1^2$ | $\frac{1}{2} \frac{g}{h} C_1^2\omega_1^3$ |
| B and E. | $\frac{4}{9} \{(n+1)^{3/2}-n^{3/2}\}^2 C_1^2$ | $\frac{4}{9} \{(n+1)^{3/2}-n^{3/2}\}^2 \frac{g}{h} C_1^2\omega_1^3$ | $\frac{4}{9} C_1^2$ | $\frac{4}{9} \frac{g}{h} C_1^2\omega_1^3$ |
| C and F. | $\frac{1}{\epsilon} \frac{(n+1)^{n+1}}{n^n} C_1^2$ | $\frac{1}{\epsilon} \frac{(n+1)^{n+1}}{n^n} \frac{g}{h} C_1^2\omega_1^3$ | $\frac{1}{\epsilon} C_1^2$ | $\frac{1}{\epsilon} \frac{g}{h} C_1^2\omega_1^3$ |

2. *The Rigid Dipole Rotator.*

| Method of Average. | $Q^2$ $n+1\to n$. | $A^n_{n+1}$. | $Q^2$ $1\to 0$. | $A_1{}^0$. |
|---|---|---|---|---|
| A, B, and C. | $C_1^2$ | $2\times\frac{(2n+1)^3}{8} \frac{g}{h} C_1^2\omega_1^3$ | $C_1^2$ | $2\times\frac{1}{8} \frac{g}{h} C_1^2\omega_1^3$ |
| D. | $\frac{16}{5} \frac{(n+1)^5-n^5}{(2n+1)^4} C_1^2$ | $2\times\frac{2}{5} \frac{(n+1)^5-n^5}{2n+1} \frac{g}{h} C_1^2\omega_1^3$ | $\frac{16}{5} C_1^2$ | $2\times\frac{2}{5} \frac{g}{h} C_1^2\omega_1^3$ |
| E. | $\frac{16}{9} \frac{\{(n+1)^3-n^3\}^2}{(2n+1)^4} C_1^2$ | $2\times\frac{2}{9} \frac{\{(n+1)^3-n^3\}^2}{2n+1} \frac{g}{h} C_1^2\omega_1^5$ | $\frac{16}{9} C_1^2$ | $2\times\frac{2}{9} \frac{g}{h} C_1^2\omega_1^3$ |
| F. | $\frac{16}{\epsilon^4} \frac{(n+1)^{4(n+1)}}{n^{4n}(2n+1)^4} C_1^2$ | $2\times\frac{2}{\epsilon^4} \frac{(n+1)^{4(n+1)}}{n^{4n}(2n+1)} \frac{g}{h} C_1^2\omega_1^3$ | $\frac{16}{\epsilon^4} C_1^2$ | $2\times\frac{2}{\epsilon^4} \frac{g}{h} C_1^2\omega_1^3$ |

3. *The Simplified Hydrogen Atom (Electron in Circular Orbit).*

| Method of Average. | $Q^2$ $n+1\to n$. | $A^n_{n+1}$. | $Q^2$ $1\to 0$. | $A_1{}^0$. |
|---|---|---|---|---|
| A. | $\frac{1}{5}\{(n+1)^5-n^5\}C_1{}^2$ | $2\times\frac{1}{40}\frac{\{(n+1)^5-n^5\}(2n+1)^3}{n^6(n+1)^6}\frac{g}{h}C_1{}^2\omega_1{}^3$ | $\frac{1}{5}C_1{}^2$ | $\infty$ |
| B. | $\frac{1}{9}\{(n+1)^3-n^3\}^2C_1{}^2$ | $2\times\frac{1}{72}\frac{\{(n+1)^3-n^3\}^2(2n+1)^3}{n^6(n+1)^6}\frac{g}{h}C_1{}^2\omega_1{}^3$ | $\frac{1}{9}C_1{}^2$ | $\infty$ |
| C. | $\frac{1}{\epsilon^4}\frac{(n+1)^{4(n+1)}}{n^{4n}}C_1{}^2$ | $2\times\frac{1}{8\epsilon^4}\frac{(n+1)^{4n-2}(2n+1)^3}{n^{4n+6}}\frac{g}{h}C_1{}^2\omega_1{}^3$ | $\frac{1}{\epsilon^4}C_1{}^2$ | $\infty$ |
| D. | $\frac{16}{7}\frac{n(n+1)\{(n+1)^7-n^7\}}{(2n+1)^4}C_1{}^2$ | $2\times\frac{2}{7}\frac{\{(n+1)^7-n^7\}}{n^5(n+1)^5(2n+1)}\frac{g}{h}C_1{}^2\omega_1{}^3$ | 0 | $\infty$ |
| E. | $\frac{16}{9}\frac{n^2(n+1)^2\{(n+1)^3-n^3\}^2}{(2n+1)^4}C_1{}^2$ | $2\times\frac{2}{9}\frac{\{(n+1)^3-n^3\}^2}{n^4(n+1)^4(2n+1)}\frac{g}{h}C_1{}^2\omega_1{}^3$ | 0 | $\infty$ |
| F. | $16\,\epsilon^8\frac{n^{8(n+1)}}{(n+1)^{8n}(2n+1)^4}C_1{}^2$ | $2\times 2\epsilon^8\frac{n^{8n+2}}{(n+1)^{8n+6}(2n+1)}\frac{g}{h}C_1{}^2\omega_1{}^3$ | 0 | 0 |

## IV. Discussion of Results.

It is a matter of interest that all the methods of averaging, with the exception of F, lead to an infinite probability that an electron in the innermost orbit of the hydrogen atom would fall during unit time into the nucleus. On the other hand, method F leads to what we must regard as experimentally correct—namely zero for the chance that the electron would fall into the nucleus *.

Although this result would seem somewhat strong evidence against all except method F, attention should be called to the fact that the treatment given can hardly be regarded as absolutely exact, since it assumes that the inverse square law of attraction would hold down to zero distance between nucleus and electron—an assumption which does not agree with the finite mass and hence finite energy of the hydrogen atom. Furthermore, it should be noted that Hoyt's results seemed unfavourable to method F. Further work on the problem is much to be desired †.

Norman Bridge Laboratory of Physics,
Pasadena, California.
July 24th, 1924.

* If we are permitted to trespass for a moment in a field of visionary speculation, it is interesting to note that although method F leads to zero for the chance that an electron would fall into the nucleus during unit time, it leads to a finite value, $2\epsilon^8 y C_1^2 \omega_1^4$, for the rate at which energy would be emitted by hydrogen atoms in the first quantum orbit, since the vanishingly small number of transitions is multiplied by the infinite energy emitted in each transition. This result might have astrophysical significance in connexion with the difficulty of discovering a sufficient source for the energy emitted by some of the stars. The falling of an electron into the hydrogen nucleus would presumably lead to the destruction of the atom with the production of an amount of radiant energy equivalent to its mass. The fact that our treatment would make the energy emitted infinite when an electron falls into the nucleus may not seriously affect the approximate correctness of our figure for the rate of energy emission.

† The writer wishes to express his thanks to Mr. I. S. Bowen for his kindness in reading the manuscript and checking the calculations involved in the above article. Mr. Bowen has also made an interesting generalization as to the results of different methods of averaging. If we take $\log(Q\nu^r)=\int_0^1 \log(C\omega^r)d\lambda$, then it can be shown that the condition $x>1{\cdot}5$ will make the probability zero that an electron in the first orbit will fall into the nucleus. If we take $Q^x\nu^y=\int_0^1 C^x\omega^y d\lambda$, then we may distinguish two cases. When $2x+1>3y$, we must have $3x<2y$ in order to make the above probability zero. When $2x+1<3y$, we must have $x+y<1$ to make the probability zero.

XIV. *The Period of Mesothorium* 2. *By* W. P. Widdowson, *B.A., Scholar of Christ Church, Oxford, and* A. S. Russell, *M.A., D.Sc., Dr. Lee's Reader in Chemistry* *.

MESOTHORIUM 2 was discovered in 1908 by O. Hahn †, who found its half-value period to be 6·2 hours. McCoy and Viol ‡ found the value 6·14 hours. Recently in an unsuccessful attempt to isolate a second product of mesothorium 1, corresponding to mesothorium 2 as uranium Z does to uranium $X_2$, we made several determinations of mesothorium 2's period. Our mean value is 5·95 hours, which is four per cent. smaller than Hahn's.

In chemical properties mesothorium 2 (atomic number 89) resembles the rare earths and is usually described as being similar to lanthanum. When fractionated along with the rare earths its isotope actinium appears to be concentrated in the neodymium and samarium fractions § so that they are homologues of the missing element of atomic number 61. In qualitative analysis mesothorium 2 is precipitated as hydroxide by ammonia, but quantitatively only in absence of a moderate quantity of ammonium salts. Its parent substance, mesothorium 1, when very concentrated contains radium and therefore also products both of the thorium and the radium series. The separation of mesothorium 2 from mesothorium 1 (atomic number 88) has consequently to be carried out in the presence of products of atomic numbers 90 (radiothorium), 88 (radium, thorium X), 86 (radon, thoron), 84 (radium A, thorium A, etc.), 83 (radium C, thorium C), and 82 (radium B, thorium B). There is, however, no difficulty in this.

The procedure we found best was as follows:—First we precipitated products of atomic numbers 90, 89, 84, 83, and 82 from those of atomic number 88 by dissolving a preparation of mesothorium bromide, equal in $\gamma$-ray activity to 6 mgrms. $RaBr_2$, in water, adding to the solution another containing the nitrates of lanthanum, thorium, and barium, and precipitating with carbonate-free ammonia. Mesothorium and its isotopes alone remain in solution with the barium. This solution was then acidified and set aside for a time. It lost radon and thoron freely so that their products soon decayed to a negligible concentration. After a month

* Communicated by the Authors.

† O. Hahn, *Phys. Zeit.* ix. p. 246 (1908).

‡ H. N. McCoy & C. H. Viol, Phil. Mag. xxv. p. 333 (1913).

§ A. Debierne, *Compt. rend.* cxxxix. p. 538 (1904).

the concentration of thorium X had fallen to about 0·004 of its original value, mesothorium 2 had attained its equilibrium amount, and a small quantity of radiothorium had grown. A small quantity of thorium, free from all products but radiothorium, was then added, and precipitated from solution by carbonate-free ammonia. This removed all radioactive products from the solution except isotopes of mesothorium 1. The filtrate was acidified, a small quantity of thorium added to it, and half-an-hour later the thorium was precipitated by ammonia. This thorium precipitate was found to contain mesothorium 2 only which had grown in the interval, for the $\beta$-ray activity decayed exponentially with a period of about six hours, and the residual activity was very small.

Several determinations of the half-value period were carried out by freeing the active solution from mesothorium 2, allowing this product to accumulate for times which varied from seven minutes to twenty-four hours, and precipitating it along with thorium with ammonia. (Thorium was used instead of lanthanum as a nucleus for the active material because it is precipitated by ammonia even in presence of a large amount of ammonium salts, and in the active solution employed these were accumulating. It was prepared free from all its products except radiothorium by adding to a solution of it, a solution of lead, bismuth, lanthanum, and barium nitrates, almost removing the acidity by ammonia, and precipitating the thorium with excess of a solution of *m*-nitrobenzoic acid. The thorium so precipitated is entirely free from all its products.) The thorium precipitate containing the mesothorium 2 was washed, dried, ignited, and its activity observed in a $\beta$-ray electroscope. These observations occupied about five minutes. In none of the preparations was any initial abnormal decay observed ; the active material decayed exponentially to zero. In seven of the best experiments the following values of the half-value period in minutes were obtained :—360, 352, 359, 361, 356, 354, and 356. The mean value is 357 minutes or 5·95 hours. The period of average life is consequently 8·58 hours, and the disintegration constant $\lambda$ is 0·116 per hour. This value of the half-value period is probably within 3 minutes of the true value and is 15 minutes less than Hahn's.

The experiments described show that there is no appreciable quantity of another product of mesothorium 1, isotopic with mesothorium 2, if such has a period between 3 minutes and 6 hours. For a hypothetical product with these properties would become concentrated in mesothorium 2 by an operation which removed them entirely from solution and then allowed

them to grow for an interval of time which is of the same order as the period of the more quickly-changing product. This product should then be revealed by an initially abnormal decay of the separated products. For example, supposing that the hypothetical product has a period of 7 minutes, and that in equilibrium it emits one $\beta$-particle for every 500 $\beta$-particles emitted by mesothorium 2. After both have been removed from a solution of mesothorium 1 and allowed to grow for 7 minutes, the ratio is raised from 1 : 500 to 1 : 10. Seven minutes later when these products have been separated from the solution and are ready for measurement in the electroscope the ratio has fallen to 1 : 20, but this is high enough to allow the presence of the minor but more quickly-changing constituent to be shown by the measurements of decay. The closer the periods of the isotopic products, the greater must be the proportion of the minor constituent in the mixture to make detection possible. In none of the experiments which we carried out was any trace of a minor isotope revealed. If, however, such an isotope had a period of about 2 minutes or less it would have escaped us unless its concentration was considerable, for 5 or 6 minutes was necessary to filter off the products and prepare them for the electroscope.

If the hypothetical product had a period considerably larger than that of mesothorium 2 it would be found in appreciable concentration only in the first precipitate of lanthanum and thorium hydroxides from the original solution of mesothorium. This precipitate was dissolved in acid, a solution of lead and barium nitrates added to it, and the lanthanum separated from the lead by sulphuretted hydrogen, from the barium by carbonate-free ammonia, and from the thorium by *m*-nitrobenzoic acid. The intensely active precipitate was then allowed to decay for several days until mesothorium 2 had attained a negligible concentration, and examined from time to time to see if a long-lived isotope was present. A small quantity of a $\beta$-ray body remained, but this proved to be, by its emanating power and period, thorium X, doubtless either adsorbed by the lanthanum or precipitated in small amount by ammonia in presence of air. This substance was dissolved in acid, a solution of barium nitrate added to it, and carbonate-free ammonia added in excess. These operations were repeated. The concentration of thorium X in the precipitated lanthanum was then found to be negligible. There appears then to be no appreciable amount of a second product of mesothorium 1 with a longer period than that of mesothorium 2 in the preparation

with which we worked. Our preparation was one that had a $\gamma$-ray activity equal to that of 6 milligrams of radium bromide. Naturally before such a question can be settled more active preparations must be examined.

*Summary.*

1. A strong source of mesothorium was examined to see if it gave a branch product isotopic with mesothorium 2. It is found that no second product with a period smaller than that of mesothorium 2 exists in any detectable amount. Preliminary experiments indicate that the existence of a product with a greater period is also unlikely.

2. The half-value period of mesothorium 2 has been measured accurately. It is 5·95 hours. This value is probably less than one per cent. in error. It is four per cent. smaller than Hahn's original value, 6·2 hours. The new value of the disintegration constant is 0·116 per hour.

A grant from the Government Grant Committee of the Royal Society kindly made to one of us was used in purchasing materials for this piece of work.

Dr. Lee's Laboratory,
Christ Church, Oxford.
October 1924.

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XV. *Note on the Poynting-Flux in a Gradually Varying Dielectric.* *By* J. A. Wilcken, *B.Sc., Ph.D., Lecturer in Electrical Engineering at Armstrong College, Newcastle-upon-Tyne* *.

In connexion with the transmission of energy along high-tension cables the problem has arisen of the influence of grading, or variation of the dielectric constant of the insulating material, on the Poynting-Flux.

To approach the problem, we consider a plane stratified medium, in which the dielectric constant $\epsilon$, and the permeability $\mu$, are real continuous functions of $x$, and take the medium bounded to the left by the plane $x=0$. A train of plane waves impinges on the boundary, coming from $x<0$, and we consider first the case of polarization in the incidence-plane $xy$.

Maxwell's equations, connecting the components X, Y, Z,

* Communicated by the Author.

of the electric force, and $\alpha$, $\beta$, $\gamma$ of the magnetic force, become

$$\left.\begin{aligned} \epsilon\frac{\partial Z}{\partial t} &= \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \\ -\mu\frac{\partial \alpha}{\partial t} &= \frac{\partial Z}{\partial y} \\ \mu\frac{\partial \beta}{\partial t} &= \frac{\partial Z}{\partial x} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating the first equation, we find

$$\epsilon\frac{\partial^2 Z}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{1}{\mu}\frac{\partial Z}{\partial x}\right) + \frac{1}{\mu}\frac{\partial^2 Z}{\partial y^2},$$

and substituting $Z = \phi e^{i(pt+\psi)}$, $\phi$ and $\psi$ being real functions of $x$ and $y$, this equation splits up into the two real equations,

$$\frac{1}{\phi}\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\} - \frac{1}{\mu\phi}\frac{\partial \mu}{\partial x}\frac{\partial \phi}{\partial x} - \left\{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 - p^2\mu\epsilon\right\} = 0, \quad . \quad (2)$$

$$\frac{\partial}{\partial x}\left(\phi^2\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi^2\frac{\partial \psi}{\partial y}\right) - \frac{1}{\mu}\phi^2\frac{\partial \mu}{\partial x}\frac{\partial \psi}{\partial x} = 0. \quad . \quad (3)$$

When $\phi$ and $\psi$ have been found, determining Z, $\alpha$ and $\beta$ can be obtained from the equation (1), viz.

$$\alpha = -\frac{1}{\mu}\int\frac{\partial Z}{\partial y}dt$$

$$= -\frac{1}{p\mu}\left(\phi\frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}\right)e^{i(pt+\psi)}$$

$$\beta = \frac{1}{p\mu}\left(\phi\frac{\partial \psi}{\partial x} - i\frac{\partial \phi}{\partial x}\right)e^{i(pt+\psi)},$$

and the field is defined by the real vector-components,

$$\left.\begin{aligned} Z &= \phi\cos\theta \\ \alpha &= a\cos(\theta - \xi) \\ \beta &= b\cos(\theta - \eta) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where
$$\left.\begin{aligned} a &= -\frac{1}{p\mu}\left\{\phi^2\left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}^{1/2} \\ b &= \frac{1}{p\mu}\left\{\phi^2\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right\}^{1/2} \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

$$\left.\begin{aligned} \xi &= \tan^{-1}\left\{\frac{\partial\phi}{\partial y}\Big/\phi\frac{\partial\psi}{\partial y}\right\} \\ \eta &= \tan^{-1}\left\{\frac{\partial\phi}{\partial x}\Big/\phi\frac{\partial\psi}{\partial x}\right\} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\theta = pt + \psi.$$

The physical interpretation of this result is that the gradual variation of the dielectric causes the magnetic force to get out of step with the electric force ; but it should be noted that this effect is not accumulative, and depends, at any point, only on the local variation at that point.

Further, the magnetic force is slightly curved, as is evident from the difference in the time-lags $\xi$ and $\eta$ ; in fact, the wave ceases to be plane.

In what follows, we shall take the variation of the dielectric to be "slow," as compared with the wave-length, so that the amplitude $\phi$ is practically constant over a considerable range. It would appear that only in this case are we justified in speaking of a "wave" as represented by the function $\phi e^{i(pt+\psi)}$. The magnetic and electric forces will now be practically at right angles, swinging in the wave-fronts of constant phase $\psi = \text{const.}$, and the direction-cosines of the wave-normal at any point will be proportional to $\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}$ respectively. An approximate solution of this problem has been discussed elsewhere *.

Evaluating the radiation-flux, according to Poynting, taking the time-average over a complete period, we get

$$\begin{aligned} \mathrm{P} &= \frac{c}{4\pi}\cdot\frac{1}{2\pi}\int_0^{2\pi}(-\mathrm{Z}\beta)\,dt \\ &= -\frac{c}{4\pi}\cdot\frac{1}{2\pi}\int_0^{2\pi}\phi\cos(pt+\psi)\cdot\frac{1}{p\mu}\left\{\phi\frac{\partial\psi}{\partial x}\cos(pt+\psi)\right. \\ &\qquad\qquad \left. + \frac{\partial\phi}{\partial x}\sin(pt+\psi)\right\}dt \\ &= -\frac{c}{8\pi p^2\mu}\phi^2\frac{\partial\psi}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7) \end{aligned}$$

* Proc. Camb. Phil. Soc. xxii. pt. 2, pp. 124–131 (1924).

$$Q = \frac{c}{4\pi} \cdot \frac{1}{2\pi} \int_0^{2\pi} Z\alpha \, dt$$

$$= -\frac{c}{8\pi p^2 \mu} \phi^2 \frac{\partial \psi}{\partial y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It is seen that the radiation is propagated along the wave-normal, as in a homogeneous medium.

Moreover, we have, on account of equation (3),

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in accordance with the non-absorbing character of the medium, and this result is not dependent on the slow variation of quality.

It should also be noted, that in the case of normal incidence, which is most important from our point of view, the equations contain only one variable, $x$, and there is complete symmetry around the ray, which is then strictly normal to the plane in which the electric and magnetic forces are swinging at right angles to one another. The time-lag between the vectors will, however, persist.

In case of polarization in the $xz$-plane, the fundamental equations become

$$\epsilon \frac{\partial X}{\partial t} = \frac{\partial \gamma}{\partial y}$$

$$\epsilon \frac{\partial Y}{\partial t} = -\frac{\partial \gamma}{\partial x}$$

$$-\mu \frac{\partial \gamma}{\partial t} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y},$$

which are seen to be completely analogous to the equations (1), leading to similar results.

## XVI. *On Orographic Precipitation.* *By* Suminosuke Ono *.

### § 1. *Introduction.*

It is a well-known fact in meteorology that the amount of precipitation depends on orographic conditions. We have many reports in which statistics show that the amount of precipitation increases with altitude in the mountain region near a sea-coast. The point of maximum precipitation, however, is not the summit of the mountain. The small rainfall of the basins of Bohemia and Hungary surrounded by the enclosing mountains of Germany and Austria, on which the rainfall is comparatively heavy, is a well-known fact. In Japan, the province of Shinano is also a region of small precipitation surrounded by mountains with somewhat heavier precipitation. These phenomena show evidently that orographic precipitation decreases extremely behind a mountain. A theoretical explanation of these phenomena has been given by Prof. F. Pockels †. His calculation shows that under certain assumptions the amount of precipitation increases with the altitude of the land, tending to a maximum at the point at which the inclination is a maximum, and vanishing at the summit of the mountain. This solution must of course be only a first approximation. In the present paper a more general solution, which can be applied to the region behind the mountain, is given; the numerical comparison being taken with the observed values of snow in Yamagata province, which lies in the north-eastern part of Honshu in the Japanese Islands.

### § 2. *Stream Function.*

Though the actual cause of meteorological precipitation may be very complicated, there is no doubt that the precipitation is directly related to the vertical currents of the atmosphere. The real motion of atmospheric air is actually very complicated; it is not steady; it whirls everywhere under the action of viscosity: but our observed data are actually statistical, and the theoretical results which are calculated under very simplified conditions may illustrate many phenomena in the atmosphere satisfactorily.

* Communicated by Sir Napier Shaw, F.R.S. This paper will be submitted to the publishing committee of papers to be dedicated to Professor Nagaoka to commemorate the twenty-fifth anniversary of his professorship.

† Pockels, *Ann. d. Physik*, (4) iv. pp. 459–480 (1901).

As is generally done, I shall now assume the current of air to be steady and continuous, free from whirls and friction, and, moreover, all conditions to be two-dimensional, *i. e.* the current is assumed to flow everywhere parallel to a definite vertical plane.

If we designate by $x$ and $z$ the horizontal and vertical co-ordinates respectively, and by $\phi$ the velocity potential, which may exist in consequence of the conditions assumed above, then the horizontal and the vertical components of the velocity $u$ and $v$ are

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial z}.$$

As is well known, the hydrodynamical equation follows

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial x} = 0,$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{\rho}\frac{\partial \rho}{\partial x}\frac{\partial \phi}{\partial x} + \frac{1}{\rho}\frac{\partial \rho}{\partial z}\frac{\partial \phi}{\partial z} = 0, \quad . \quad . \quad (1)$$

in which $\rho$ is the density of air.

In this equation the term $\frac{\partial \rho}{\partial x}$ may be assumed to be zero, and $\frac{\partial \rho}{\partial z}$ takes various functional forms corresponding with the assumed conditions. In this paper the following four cases are discussed, but in every case, terms of density variation due to the hydrodynamical effect are neglected.

*Case A.—Constant density.*

This case is applicable when the consideration is restricted to a comparatively small vertical height.

Since in this case $\frac{\partial \rho}{\partial z} = 0$, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The solution of this partial differential equation is

$$\sum_j \{\phi_j(x+iz) + \phi_j(x-iz)\} = 0, \quad . \quad . \quad . \quad (3)$$

$\phi_j$ being an arbitrary function of a complex variable. In many practical cases it will be convenient for calculation if trigonometrical and exponential functions are used.

*Case B.—Isothermal.*

In this case, by Boyle's law, the density of air is directly proportional to the pressure, and the relation between the pressure $p$ and the vertical coordinate $z$ is given by a logarithmic relation, *viz.*,

$$\log\frac{p_0}{p} = \log\frac{\rho_0}{\rho} = q_1 z.$$

In this equation $q_1$ is $\frac{g}{RT}$; $g$, R, and T denoting the acceleration due to gravity, the gas constant, and the absolute temperature of air respectively. Consequently

$$\frac{1}{\rho}\frac{\partial\rho}{\partial z} = -q_1$$

is a constant, and the equation (1) becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - q_1\frac{\partial\phi}{\partial z} = 0. \quad . \quad . \quad . \quad . \quad (4)$$

Pockels * gave a particular solution of this equation with a series containing trigonometrical terms may correspond with Fourier's series representing an arbitrary orographic form. But we may have a more general solution of this equation as follows:—

Changing the coordinates $x$, $z$ into the cylindrical coordinates $r$, $\theta$ (*i. e.*, $x = r\cos\theta$, $z = r\sin\theta$) and substituting $e^{\frac{1}{2}q_1 x + ni\theta}\mathrm{K}(r)$ for $\phi$ in the last equation, we may have

$$\frac{\partial^2 \mathrm{K}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{K}}{\partial r} - \left\{\left(\frac{q_1}{2}\right)^2 + \frac{n^2}{r^2}\right\}\mathrm{K} = 0. \quad . \quad . \quad (5)$$

Since this equation is satisfied with the cylindrical function of imaginary argument, we have the solution of equation (4)

$$\phi = \mathrm{G}(z) \,.\, x + e^{\frac{1}{2}q_1 z}\sum_j\sum_n (\mathrm{A}_{nj}\cos n\theta_j + \mathrm{B}_{nj}\sin n\theta_j)\mathrm{K}_n(\tfrac{1}{2}q_1 r_j i), \quad . \quad . \quad . \quad (6)$$

in which $\mathrm{K}_n(\frac{1}{2}q_1 r_j i)$ denotes the cylindrical function, and

$$\mathrm{G}(z) \,.\, x = \left\{a + \frac{c}{q_1}e^{q_1 z}\right\} x$$

is a particular solution, $a$, $c$, $\mathrm{A}_{nj}$, and $\mathrm{B}_{nj}$ being constants.

* Pockels, *loc. cit.*

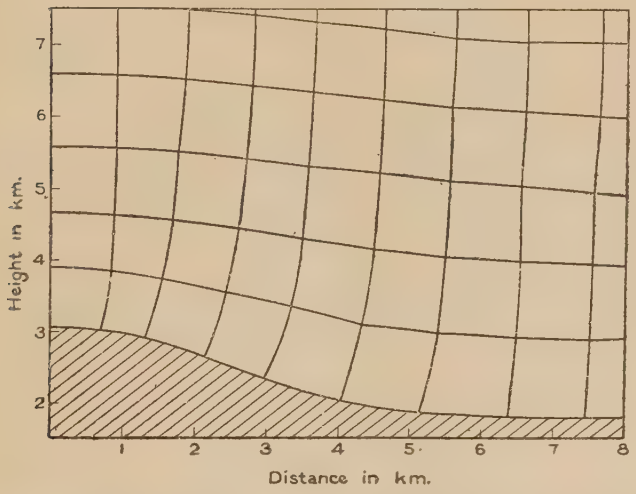
An example of equipotential surfaces and stream lines calculated by this formula is shown in fig. 1, in which the following values of the constants are used :—

$a = 0{\cdot}01$ km./sec., $\quad A_1 = -0{\cdot}001$ km.$^2$/sec.,

$q_1 = 0{\cdot}125$/km., and all other constants vanish; *viz.*

$$\phi = ax + Ae^{\frac{1}{2}q_1 z}\cos\theta\, K_1(\tfrac{1}{2}q_1 ri).$$

Fig. 1.




[Equipotential surfaces and stream lines in an isothermal distribution.

*Case C.—Adiabatic equilibrium.*

In this case we have

$$\frac{p}{\rho} = RT, \qquad \frac{1}{p}\frac{\partial p}{\partial z} = -\frac{g}{RT}$$

and

$$\frac{1}{p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial \rho}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} = -\frac{1}{T}\frac{g}{JC_p},$$

in which J and $C_p$ denote the mechanical equivalent of heat and the specific heat of air under constant pressure respectively, the other symbols being the same as those used in the former cases.

In virtue of the last equation, we have

$$T = T_0 - \frac{g}{C_p J} z = T_0(1 - \kappa z), \quad . \quad . \quad . \quad . \quad (7)$$

in which $T_0$ is the temperature at the point $z = 0$, and $\frac{g}{JC_p}T_0$ is designated by $\kappa$. Consequently we have

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{g}{RT} + \frac{g}{JC_p}\frac{1}{T}$$

$$= -(q_1 - \kappa)\frac{1}{1 - \kappa z},$$

$\frac{g}{RT_0}$ being designated by $q_1$. And equation (1) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - (q_1 - \kappa)\frac{1}{1 - \kappa z}\frac{\partial \phi}{\partial z} = 0. \quad . \quad . \quad (8)$$

The solution of this equation may be considered as a special case, which is discussed in the next paragraph.

*Case D.—Pseudoadiabatic equilibrium.*

This is a case where a special state of equilibrium holds, water or ice precipitating steadily in the air. In this case the relation between the temperature and the height is *

$$\frac{\partial T}{\partial z} = -\frac{g\left(\frac{1}{J} + \frac{s\omega}{RT}\right)}{C_p + s\omega \frac{1}{b}\frac{\partial b}{\partial T}}. \quad . \quad . \quad . \quad . \quad (9)$$

The notation in this equation is the same as in the former cases, except $s$, $b$, and $\omega$, by which the latent heat of vaporization (from liquid phase or solid phase), the saturated vapour-pressure, and the mass of the saturated water vapour in the unit mass of air are designated respectively. And in these $\omega$ is given by the following formula,

$$\omega = 0{\cdot}622\,\frac{b}{p},$$

in which 0·622 is the ratio of the molecular weight of water vapour to that of air. Practically $s$ in the above equation may be considered as a constant, while $b$ is a function of the temperature. There are many formulæ by which the relation between the vapour-pressure and the temperature is

* Humphreys, 'Physics of the Air,' p. 34 (1920).

expressed. But among them is Regnault's formula *, which has been checked by various authors with their measurements, *viz.*:

$$\log b = c + c'\beta^t + c''\gamma^t;$$

$c$, $c'$, $c''$, $\beta$, and $\gamma$ are constants, and $t$ is the temperature. Since among these $\beta$ and $\gamma$ are very near to unity, expanding $\beta^t$ and $\gamma^t$ in a series of ascending powers of $t$ and neglecting terms of higher order than the first, the logarithmic value of $b$ can be expressed by a linear function of $t$, *viz.*:

$$\log b = c + c' + c'' + \{c'(\beta-1) + c''(\gamma-1)\}t.$$

Or, designating the value of $\{c'(\beta-1) + c''(\gamma-1)\}$ by $l$, we may write

$$b = b_0 e^{l(T-T_0)},$$

where T is the absolute temperature, and $b_0$ denotes the value of $b$ when T is equal to $T_0$, the temperature at the point $z=0$.

For the value of $p$, since $\frac{1}{p}\frac{\partial p}{\partial z} = -\frac{g}{RT}$, we have

$$p = p_0 e^{-\int_0^z \frac{g}{RT} dz}.$$

And the equation (9) becomes

$$\frac{\partial T}{\partial z} = -\frac{g\left\{\frac{1}{J} + \frac{0{\cdot}622\, sb_0}{RT p_0} e^{l(T-T_0)+\int_0^z \frac{g}{RT} dz}\right\}}{C_p + 0{\cdot}622 \frac{sb_0 l}{p_0} e^{l(T-T_0)+\int_0^z \frac{g}{RT} dz}} \quad . \quad . \quad (9')$$

But since $\frac{s\omega}{RT}$ and $s\omega l$ are small compared with $\frac{1}{J}$ and $C_p$ respectively, it is evident that the first approximate solution must be given in the form

$$T = T_0(1-\kappa' z),$$

where

$$\kappa' = \frac{\frac{g}{T_0}\left\{\frac{1}{J} + \frac{0{\cdot}622\, sb_0}{RT p_0} e^{l(T_1-T_0)+\int_0^{z_1} \frac{g}{RT_1} dz}\right\}}{C_p + 0{\cdot}622 \frac{sb_0 l}{p_0} e^{l(T_1-T_0)+\int_0^{z_1} \frac{g}{RT_1} dz}}$$

$T_1$ and $z_1$ being the mean temperature and the mean height respectively. To obtain a more approximate solution, if we

* Regnault, *Rél. des exp.*

put $T_0(1-\kappa' z)$ for T in the right-hand side of the equation (9′), there results

$$T = T_0 - \int_0^z \frac{g\left\{\frac{1}{J} + \frac{0\cdot622}{RT_0}\frac{sb_0}{p_0} e^{lT_0\kappa' z}(1-\kappa' z)^{-1-\frac{g}{RT_0\kappa'}}\right\}}{C_p + 0\cdot622\frac{sb_0 l}{p_0} e^{lT_0\kappa' z}(1-\kappa' z)^{-\frac{g}{RT_0\kappa'}}} dz, \quad \ldots \quad (10)$$

and this may be expressed in a power series of $z$,

$$T = T_0(1-\kappa' z+\kappa'' z^2+ \ldots). \quad \ldots \quad (10')$$

On the other hand,

$$\begin{aligned} -\frac{1}{\rho}\frac{\partial \rho}{\partial z} &= -\frac{1}{p}\frac{\partial p}{\partial z} + \frac{1}{T}\frac{\partial T}{\partial z} \\ &= \frac{1}{T}\left\{\frac{g}{R} + \frac{\partial T}{\partial z}\right\} \\ &= \frac{\frac{g}{RT_0} - \kappa' + 2\kappa'' z + \ldots}{1-\kappa' z+\kappa'' z^2+ \ldots}, \\ &= Q(z), \quad \text{say.} \end{aligned}$$

And we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - Q(z)\frac{\partial p}{\partial z} = 0. \quad \ldots \quad (11)$$

Or, if we neglect the terms of higher order and put

$$\frac{g}{RT_0} - \kappa' = q',$$

it follows that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{q'}{1-\kappa' z}\frac{\partial \phi}{\partial z} = 0. \quad \ldots \quad (11')$$

This equation is quite similar to equation (8) in the case of the adiabatic equilibrium.

To solve this equation, if we put

$$\phi = (1-\kappa' z)^{-\frac{q'}{2\kappa'}}\psi,$$

we have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} - \left(\frac{q'^2}{4} - \frac{q'\kappa'}{2}\right)\frac{\psi}{(1-\kappa' z)^2} = 0.$$

In this equation, since $\kappa'$ is very small, we can put

$$\frac{1}{(1-\kappa' z)^2} = e^{2\kappa' z},$$

and denoting the value $\frac{q'^2}{4} - \frac{q'\kappa'}{2}$ by $\frac{1}{4}q^2$,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{q^2}{4} e^{2\kappa' z}\psi = 0.$$

Now changing the variables $x$, $z$ to $x'$, $z'$, which are defined by the following equations:

$$\left.\begin{aligned} z' + ix' &= \frac{1}{\kappa'}\left\{e^{\kappa'(z+ix)} - 1\right\} \\ z' - ix' &= \frac{1}{\kappa'}\left\{e^{\kappa'(z-ix)} - 1\right\} \end{aligned}\right\}, \quad . \quad . \quad . \quad (12)$$

the equation (11′) becomes

$$\frac{\partial^2\psi}{\partial x'^2} + \frac{\partial^2\psi}{\partial z'^2} - \frac{q^2}{4}\psi = 0,$$

which is quite similar to equation (4). Thus the solution of equation (11′) must be

$$\phi = \mathrm{G}(z) \,.\, x + (1-\kappa' z)^{-\frac{q'}{2\kappa'}} \sum_j \sum_n (\mathrm{A}_{nj} \cos n\theta_j' + \mathrm{B}_{nj} \sin n\theta_j') \mathrm{K}_n\left(\frac{q}{2} r_j' i\right), \quad . \quad . \quad . \quad (13)$$

where

$$\mathrm{G}(z) = a + c\int e^{\int \frac{\frac{q'}{1-\kappa' z}\, dz}{}} dz = a + \frac{c}{q' - \kappa'}(1-\kappa' z)^{-\frac{q'}{\kappa'}+1},$$

$$r'^2 = x'^2 + z'^2 = \frac{1}{\kappa'^2}\left\{e^{2\kappa' z} - e^{\kappa'(z+ix)} - e^{\kappa'(z-ix)} + 1\right\},$$

$$\theta' = \frac{1}{i}\log\frac{z' + ix'}{r'},$$

$z'$ and $x'$ being defined by equations (12).

But this solution is rather impractical; a more practical one (though it is a particular solution) is as follows:—

Equations (2), (4), (8), and (11) can be written in the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \mathrm{Q}(z)\frac{\partial\phi}{\partial z} = 0, \quad . \quad . \quad . \quad (14)$$

where $\mathrm{Q}(z)$ is a function of $z$ representing the value of

$-\frac{1}{\rho}\frac{\partial \rho}{\partial z}$, and it is evident that the function $Q(z)$ can be expressed as a power series of $z$,

$$Q(z) = q(1+\kappa_1 z+\kappa_2 z^2+ \ldots).$$

Thus if we put

$$\phi = e^{\frac{1}{2}\int^z Q(z)\,dz+F(x,z)},$$

this satisfies equation (14) when the constants $\alpha_s$, $\alpha_s'$ and $\epsilon$, $\epsilon'$ in the following expression for $F(x,z)$ are suitable.

$$F(x,z) = \alpha_0+\alpha_{1\cdot 0}x+\alpha_{1\cdot 1}z+\alpha_{2\cdot 0}x^2+\alpha_{2\,1}xz+\alpha_{2\cdot 2}z^2+ \ldots$$

$$+\epsilon\log x+\epsilon'\log z+\frac{\alpha'_{1\cdot 0}}{x}+\frac{\alpha'_{1\cdot 1}}{z}+ \ldots . \quad . \quad . \quad (15)$$

Or more simply, if we put

$$F(x,z) = \epsilon\log x+\epsilon'\log z+\alpha_0+\alpha x+\beta z+\beta' z^2+ \ldots, \quad (15')$$

as the condition for satisfying equation (15'), there follows

$$\epsilon^2-\epsilon = 0, \qquad \epsilon'^2-\epsilon' = 0,$$
$$\alpha\epsilon = 0, \qquad \beta\epsilon' = 0,$$
$$\alpha^2+\beta^2+2\beta'+4\beta'\epsilon' = \tfrac{1}{4}q^2-\tfrac{1}{2}q\kappa_1,$$
$$4\beta\beta'+6\beta''+6\beta''\epsilon' = \tfrac{1}{2}q^2\kappa_1-q\kappa_2,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Of these the first four equations are satisfied by the following four sets of values :—

(i.) $\epsilon = 1, \quad \epsilon' = 0, \quad \alpha = 0,$

(ii.) $\epsilon = \epsilon' = 1, \quad \alpha = \beta = 0,$

(iii.) $\epsilon = 0, \quad \epsilon' = 1, \quad \beta = 0,$

(iv.) $\epsilon = \epsilon' = 0.$

It is evident that the first two sets of these correspond to the particular solution

$$G(z)x = \left\{a+c\int^z e^{\int^z Q(z)\,dz}dz\right\}x,$$

and the others become the solution which may be written in a form $e^{\alpha x+\beta z}E(z)$, $E(z)$ being a function of $z$ only. Here we can take, for convenience' sake, $\alpha^2+\beta^2=\frac{1}{4}q^2$. And the solution for $\phi$ becomes

$$\phi = \left\{a+c\int^z e^{\int^z Q(z)\,dz}dz\right\}x+c^{\frac{1}{2}\int^z Q(z)\,dz}\sum_j e^{\alpha_j x+\beta_j z}E_j(z).$$

$$. \quad . \quad . \quad (16)$$

As a special case of this solution, if we put $Q(z)=q$ (a constant), $E_j(z)$ becomes constant. Designating this constant by $A_j$, it follows,

$$\phi = \left(a+\frac{c}{q}e^{qz}\right)x+\sum_j A_j e^{\alpha_j x+(\beta_j+\frac{1}{2}q)z}. \quad . \quad . \quad (16')$$

Or more practically, if we express $\alpha_j$ and $\beta_j$ in the form of a complex number

$$\alpha_j = \lambda_j+m_j i,$$

$$\beta_j+\tfrac{1}{2}q = \mu_j+n_j i,$$

taking $\alpha_j^2+\beta_j^2=\frac{1}{4}q^2$, it follows that

$$m_j^2-\lambda_j^2+n_j^2-\mu_j^2+\mu_j q = 0,$$

$$2m_j\lambda_j+2n_j\mu_j-n_j q = 0.$$

And the stream function becomes

$$\phi = \left(a+\frac{c}{q}e^{qz}\right)x+\sum_j A_j \cos(m_j x+n_j z+\theta_j)\, e^{\lambda_j x-\mu_j z}, \quad (16'')$$

in which $a$, $c$, $A_j$, and $\theta_j$ are arbitrary constants. Pockel's solution is a special case of this. In his solution, the constant $\mu_j$ in this equation is restricted to be negative. But it seems to be rather natural if it is taken as a positive constant. An example of the recent observations in Japan is shown in Table I. The figures are the mean of 14 observations which

TABLE I.

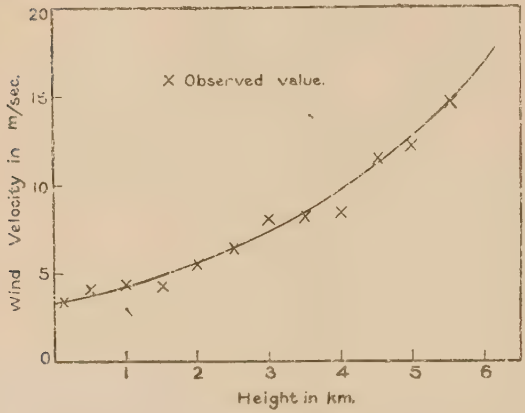
| Height. | Horizontal Wind Velocity. | Height. | Horizontal Wind Velocity. |
|---|---|---|---|
| km. | m./sec. | km. | m./sec. |
| 0·1 | 3·3 | 3·0 | 8·0 |
| 0·5 | 4·1 | 3·5 | 8·2 |
| 1·0 | 4·4 | 4·0 | 8·4 |
| 1·5 | 4·3 | 4·5 | 11·4 |
| 2·0 | 5·5 | 5·0 | 12·2 |
| 2·5 | 6·7 | 5·5 | 14·7 |

were made by Mr. S. Sekine in 1921 and 1922, by the method of pilot balloons, at the Tokorozawa Military Meteorological Station, where the orographic effects, if any,

are considered to be very small. In fig. 2 this relation between the horizontal velocity and the height is shown graphically. By these we can see that the horizontal velocity $u$ can be given by a simple exponential function. The curve in this figure represents the value calculated by a formula $Ae^{\mu z}$, the values of the constants being

$$A = 3{\cdot}27 \text{ m./sec.} \quad \text{and} \quad \mu = +0{\cdot}274/\text{km.}$$

Fig. 2.




Relation between wind velocity and height at Tokorozawa deduced from observations of pilot balloons.

## § 3. *Equation for Stream Lines.*

The constants in the above formulæ for the stream function, solved in the last paragraph, must be determined by the boundary conditions. For the purpose of determining these constants, we must have the equation for the stream line. If we denote this equation in a form $\chi=0$, the function $\chi$ must be given by the relation

$$\frac{\partial\phi}{\partial x}\frac{\partial\chi}{\partial x}+\frac{\partial\phi}{\partial z}\frac{\partial\chi}{\partial z}=0.$$

When the stream function $\phi$ is defined by equation (14),

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}-Q(z)\frac{\partial\phi}{\partial z}=0,$$

it is easily shown that the function $\chi$ must be given in the following form :

$$\chi = \int^{x} \frac{\partial \phi_1}{\partial z}\,\partial x + \mathrm{G}'(z)\,\frac{x^2}{2} - e^{\int^{z} \mathrm{Q}(z)\,dz} \int^{z} \mathrm{G}(z)\, e^{-\int^{z} \mathrm{Q}(z)\,dz}\,dz + \mathrm{N}\, e^{\int^{z} \mathrm{Q}(z)\,dz}, \quad . \quad (17)^*$$

in which N is the constant of integration and $\mathrm{G}'(z)$ is $\frac{d}{dz}(\mathrm{G}(z))$, and $\phi_1$ is $\phi - \mathrm{G}(z)x$.

* The lower limit of the integration in the first term of this equation is omitted. If we take this limit of the integration as $x=\gamma$, and express the first term as $\int_{\gamma}^{x} \frac{\partial \phi_1}{\partial z}\,\partial x$, this is equivalent to the addition of a term $\left[\int \frac{\partial \phi_1}{\partial z}\,\partial x\right]_{x=\gamma}$, which is a function of $z$ only. But in this case it is evident that another term,

$$-e^{+\int^{z} \mathrm{Q}(z)\,dz} \int^{z} \left(\frac{\partial \phi_1}{\partial x}\right)_{x=\gamma} e^{-\int^{z} \mathrm{Q}(z)\,dz}\,dz, \quad . \quad . \quad . \quad (\mathrm{A})$$

must be added at the same time, for the sake of satisfying the necessary condition which is given by equation (14).

If we put now

$$\mathrm{Z} = \int_{\gamma} \frac{\partial \phi_1}{\partial z}\,\partial x,$$

since

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} - \mathrm{Q}(z)\,\frac{\partial \phi_1}{\partial z} = 0,$$

we have

$$\mathrm{Z} = -\int^{z} \partial z \int^{\gamma} \frac{\partial^2 \phi_1}{\partial z^2}\,\partial x = \int^{z} \partial z \int^{\gamma} \left\{ \frac{\partial^2 \phi_1}{\partial x^2} - \mathrm{Q}(z)\,\frac{\partial \phi_1}{\partial z} \right\} \partial x,$$

$$= \int^{z} \partial z \left[ \left(\frac{\partial \phi_1}{\partial x}\right)_{x=\gamma} - \mathrm{Q}(z) \int^{\gamma} \left(\frac{\partial \phi_1}{\partial z}\right) \partial x \right]$$

$$= \int^{z} \partial z \left[ \left(\frac{\partial \phi_1}{\partial x}\right)_{x=\gamma} + \mathrm{Q}(z)\,\mathrm{Z} \right].$$

Solving this equation, we shall find that

$$\mathrm{Z} = e^{\int^{z} \mathrm{Q}(z)\,dz} \int^{z} \left(\frac{\partial \phi_1}{\partial x}\right)_{x=\gamma} e^{-\int \mathrm{Q}(z)\,dz}\,dz.$$

This is a term which cancels with the term shown in (A).

The following formulæ are solutions for various cases :—

(*a*) If $Q(z) = q$, a constant,

$$\chi = \int^x \left(\frac{\partial \phi_1}{\partial z}\right) \partial x + c\, e^{qz} \frac{x^2}{2} - \frac{c}{q} z\, e^{qz} + \frac{a}{q} + \mathrm{N}\, e^{qz} = 0. \quad . \quad . \quad . \quad (18)$$

(*b*) If $\phi = \mathrm{A} \cos\theta\, \mathrm{K}_1^{(1)}\left(\frac{q}{2} ri\right) e^{\frac{q}{2} z} + ax,$

$$\chi = \mathrm{A} \sin\theta\, \mathrm{K}_1^{(1)}\left(\frac{q}{2} ri\right) e^{\frac{q}{2} z} + i\, \mathrm{K}_0^{(1)}\left(\frac{q}{2} ri\right) e^{\frac{q}{2} z} + \frac{a}{q} + \mathrm{N}\, e^{qz} = 0. \quad (19)$$

If $\phi = \sum_j \mathrm{A}_j \cos\theta_j\, \mathrm{K}_1^{(1)}\left(\frac{q}{2} r_j i\right) e^{\frac{q}{2} z} + ax,$

$$\chi = \sum_j \left\{ \mathrm{A}_j \sin\theta_j\, \mathrm{K}_1^{(1)}\left(\frac{q}{2} r_j i\right) + i\, \mathrm{K}_0^{(1)}\left(\frac{q}{2} r_j i\right) \right\} e^{\frac{q}{2} z} + \frac{a}{q} + \mathrm{N}\, e^{qz} = 0. \quad (19')$$

(*c*) If $\phi = \mathrm{G}(z) \,.\, x + \sum_j e^{\alpha_j x + \beta_j z}\, \mathrm{E}_j(z),$

$$\chi = \sum_j e^{\alpha_j x + \beta_j z} \left\{ \frac{\beta_j}{\alpha_j} \mathrm{E}_j(z) + \frac{1}{\alpha_j} \mathrm{E}'(z) \right\} + \mathrm{G}'(z) \frac{e^2}{2} - e^{\int^z \mathrm{Q}(z)\, dz} \int^z \mathrm{G}(z)\, e^{-\int^z \mathrm{Q}(z)\, dz}\, dz + \mathrm{N}\, e^{\int^z \mathrm{Q}(z)\, dz} = 0. \quad (20)$$

If $\phi = ax + \sum_j \mathrm{A}_j \cos(m_j x + n_j z + \theta_j)\, e^{\lambda_j x + \mu_j z},$

$$\chi = \sum_j \mathrm{A}_j \left[ \frac{m_j n_j + \lambda_j \mu_j}{m_j^2 + \lambda_j^2} \cos(m_j x + n_j z + \theta_j) - \frac{n_j \lambda_j - m_j \mu_j}{m_j^2 + \lambda_j^2} \sin(m_j x + n_j z + \theta_j) \right] e^{\lambda_j x + \mu_j z} + \frac{a}{q} + \mathrm{N}\, e^{qz} = 0. \quad (21)$$

In this equation, if we put

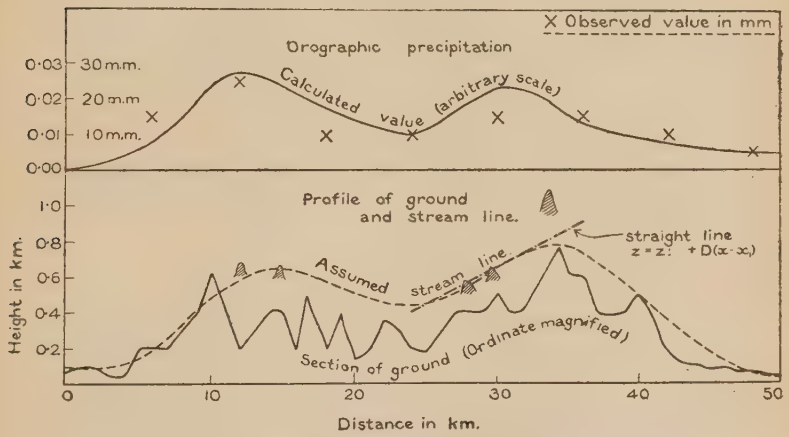
$$\mathrm{A}_j \frac{m_j n_j + \lambda_j \mu_j}{m_j^2 + \lambda_j^2} = \mathrm{C}_j \cos\alpha_j, \qquad \mathrm{A}_j \frac{n_j \lambda_j - m_j \mu_j}{m_j^2 + \lambda_j^2} = \mathrm{C}_j \sin\alpha_j,$$

we can write

$$\chi = \sum_j C_j \cos(m_j x + n_j z + \theta_j + \alpha_j) + \frac{a}{q} + N e^{qz} = 0. \quad \ldots \quad (21')$$

The dotted curve in fig. 3 is an example of the last equation representing the smoothed form of the assumed stream line, approximating to the actual profile of the ground of a local part in Yamagata Province. (The ordinate in this figure is magnified to exaggerate the irregularities of the form.) The actual section of the ground cut by a vertical plane in the direction N.W.–S.E. is shown with a full line. And the shaded forms of summits show the situation of mountains which are very near to the section.

Fig. 3.




In the case of such application it is sufficient if we take two special cases: *i. e.*,

(i.) $n_j = 0$, $\lambda_j = 0$, and $\alpha_j + \theta_j = 0$.

(ii.) $m_j = 0$ and $\alpha_j + \theta_j = 0$.

And for the profile of the ground, since the value of $z$, which represents the height of the ground, is very small compared with the other distances, we may have

$\sin n_j z = n_j z$, $\cos n_j z = 1$, $e^{\mu_j z} = 1 + \mu_j z$, and $e^{qz} = 1 + qz$.

Consequently

$$z = \frac{-\left(\frac{a}{Nq} + 1\right)\frac{1}{q} - \sum_{n=0,\,\lambda=0} \frac{C_j}{Nq} \cos m_j x - \sum_{m=0} \frac{C_j}{Nq} e^{\lambda_j x}}{1 + \sum_{n=0,\,\lambda=0} \frac{C_j}{Nq} \mu_j \cos m_j x + \sum_{m=0} \frac{C_j}{Nq} \mu_j e^{\lambda_j x}}. \quad (22)$$

By such simplification the constants are easily determined. The values in the present example are

Table II.

| $j$. | $\frac{C_j}{Nq}$. | $m_j$. | $n_j$. | $\alpha_i+\theta_j$. | $\lambda_j$ | $\mu_j$. |
|---|---|---|---|---|---|---|
| 1 ... | +0·25 | $2\pi/48$ | 0 | 0 | 0 | −0·0894 |
| 2 ... | +0·25 | $2\pi/24$ | 0 | 0 | 0 | −0·216 |
| 3 ... | +0·70 | 0 | +0·0272 | 0 | −0·0578 | +0·051 |
| 4 ... | +0·70 | 0 | +0·139 | 0 | −0·148 | +0·051 |

and

$$\tfrac{1}{2}q = \mu_3 = \mu_4 = 0{\cdot}051,$$
$$\left(\frac{a}{Nq}+1\right)\frac{1}{q} = -0{\cdot}60,$$

the unit of length being taken in kilometres.

It must be remarked that the value of N, in these, remains arbitrary.

## § 4. *Amount of Precipitation.*

It is evident that the amount of water or snow precipitated in unit time in the air-current through unit area of a horizontal plane will be given by

$$-\partial(v\omega\rho),$$

in which $\rho$ is the density of air, $v$ the vertical component of the air-current, and $\omega$ the amount of water vapour contained in a unit mass of air. Or, if we consider that the precipitation is produced within certain limits of height $h_1$ and $h_2$, the amount of the precipitation produced in a vertical column with unit area of the cross-section is

$$-\int_{h_1}^{h_2}\frac{\partial(v\omega\rho)}{\partial h}dh = v_1\omega_1\rho_1 - v_2\omega_2\rho_2, \quad . \quad . \quad (23)$$

in which $v_1$, $\omega_1$, $\rho_1$ and $v_2$, $\omega_2$, $\rho_2$ are values of $v$, $\omega$, $\rho$ at $h=h_1$ and $h=h_2$ respectively. Here we may expect that the value of $v_2\omega_2\rho_2$ is very small in most cases, because $v_2$ must be very small.

Now we have to consider how the precipitation falls through the air, and what is its distribution on the ground. It is evident that the precipitation does not fall vertically, but flows some distance with the air-current, and this distance depends upon the velocity of the air and on the size and the form of the precipitation. But the size and the form of the precipitation are not entirely constant. For the sake of convenience I shall now consider statistically the

distribution of flow of the precipitation condensed in a vertical plane perpendicular to the $x$-axis. A part of the precipitation condensed in a vertical plane $x=x'$ will fall in a plane not far from that plane $x=x'$, but some part will flow to a very distant place. Such a distribution must be considered as a function of $x$, and in ordinary cases it will have a maximum point for a certain value of $x$, and must vanish when $x=\infty$. Such a function can be expressed by a series of exponential functions * ; *i. e.*,

$$f(x) = \sum_s \eta_s e^{\eta_s(x-x')},$$

where $x > x'$ and the $\eta_s$'s are constants. In the present case the maximum point of $f(x)$ can be considered to be very near to the plane $x=x'$, and we may write very approximately $f(x)=\eta e^{-\eta(x-x')}$, and the precipitation at any point will be proportional to

$$\mathrm{W} = \int_{x_1}^{x} (v_1\omega_1\rho_1)\eta e^{-\eta(x-x')}\,dx' - \int_{x_1}^{x} (v_2\omega_2\rho_2)\eta e^{-\eta(x-x')}\,dx',$$

$x_1$ being a certain value of $x$ which shows the limit of the region producing the precipitation.

Thus, if we use the equation (16), *viz.*,

$$\phi = \mathrm{G}(z)\,.\,x + \sum_j e^{\alpha_j x+\beta_j z}\mathrm{E}_j(z),$$

we have

$$v = \frac{\partial\phi}{\partial z} = \mathrm{G}'(z)\,.\,x + \sum_j e^{\alpha_j x+\beta_j z}\{\beta_j\mathrm{E}_j(z) - \mathrm{E}'_j(z)\};$$

hence

$$\begin{aligned}\mathrm{W} = {} & \{\mathrm{G}'(z_1)\omega_1\rho_1 - \mathrm{G}'(z_2)\omega_2\rho_2\}\left\{x - x_1e^{-\eta(x-x_1)} - \frac{1}{\eta}(1-e^{-\eta(x-x_1)})\right\} \\ & + \sum_j \left\{\frac{\eta}{\alpha_j+\eta}\left[\left(\beta_j\mathrm{E}_j(z_1) + \mathrm{E}'_j(z_1)\right)\omega_1\rho_1 e^{\beta_j z_1}\right.\right. \\ & \left.\left. - \left(\beta_j\mathrm{E}_j(z_2) + \mathrm{E}'_j(z_2)\right)\omega_2\rho_2 e^{\beta_j z_2}\right]\left(e^{\alpha_j x} - e^{\alpha_j x_1}e^{-\eta(x-x_1)}\right)\right\}. \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \quad . \quad . \quad (24)\end{aligned}$$

Or, in the case of equation (16″), we have

$$\phi = \left(a + \frac{c}{q}e^{qz}\right)x + \sum_j \mathrm{A}_j\cos\left(m_jx + n_jz + \theta\right)e^{\lambda_j x+\mu_j z},$$

$$\begin{aligned}v = ce^{qz}\,.\,x + \sum_j \mathrm{A}_j\{ & \mu_j\cos\left(m_jx + n_jz + \theta_j\right) \\ & - n_j\sin\left(m_jx + n_jz + \theta_j\right)\}\,e^{\lambda_j x+\mu_j z}.\end{aligned}$$

* Some notes about this are given in the author's other paper (Journ. of Met. Soc. of Japan, 1923, p. 127).

Neglecting $v_2\omega_2\rho_2$,

$$W = ce^{qz_1}\omega_1\rho_1 \left\{ x - x_1 e^{-\eta(x-x_1)} - \frac{1}{\eta}(1-e^{\eta(x-x_1)}) \right\}$$

$$+ \sum_j \frac{A_j\omega_1\rho_1\eta}{m_j^2+(\lambda_j+\eta)^2} \{ [S_j \cos(m_j x + n_j z_1 + \theta_j) + S_j' \sin(m_j x + n_j z_1 + \theta_j)] e^{\lambda_j x + \mu_j z_1}$$

$$- [S_j \cos(m_j x_1 + n_j z_1 + \theta_j) + S_j' \sin(m_j x_1 + n_j z_1 + \theta_j)] e^{\lambda_j x_1 + \mu_j z_1} e^{-\eta(x-x_1)} \}, \quad (25)$$

where

$$S_j = \mu_j(\lambda_j+\eta) + m_j n_j,$$

$$S_j' = \mu_j m_j - n_j(\lambda_j+\eta).$$

Of course this formula is applicable to the region where $\frac{\partial z}{\partial x} > 0$.

When there are many maxima and minima in the curve of the stream line, let $x=x_1$ be the vertical plane in which the air-current begins to ascend, and $x=x_2$ the vertical plane in which the first maximum point of the stream line lies, $x=x_3$ the vertical plane in which the next minimum point lies, $x=x_4$ the vertical plane in which the second maximum point lies, and so on.

Then it is evident that

$$W_1 = \int_{x_1}^{x} (v_1\omega_1\rho_1)\eta e^{-\eta(x-x')}\,dx' \qquad \text{for} \quad x_1 < x < x_2,$$

$$W_2 = \int_{x_1}^{x_2} (v_1\omega_1\rho_1)\eta e^{-\eta(x-x')}\,dx' = W_1 e^{-\eta(x-x_2)} \quad \text{for} \quad x_2 < x < x_3,$$

$$W_3 = \int_{x_1}^{x_2} (v_1\omega_1\rho_1)\eta e^{-\eta(x-x')}\,dx' + \int_{x_3}^{x} (v_1\omega_1\rho_1) e^{-\eta(x-x')}\,dx' \qquad \text{for} \quad x_3 < x < x_4,$$

etc.

If the top of the mountain is higher than the lowest limit of the precipitating cloud, the value of $z_1$ must be considered as a function of $x$. In this case the integration will be very complicated. But for practical purposes an approximate calculation can be obtained easily. Since the maximum difference of $z$ in a stream line near to the ground is not so great compared with the other length, and terms of higher orders containing $z$ can be neglected, we may have approximately

$$v_1 = \sum_j A_j \{ -n_j \sin(m_j x + \theta_j) + \mu_j \cos(m_j + \theta_j) + (-2n_j\mu_j \sin(mj + \theta_j) + (\mu_j^2 - n_j^2)\cos(m_j x + \theta_j))z \} e^{\lambda_j x},$$

and $$\rho\omega = \rho_0\omega_0\{1-(l\kappa'\mathrm{T}_0-\kappa')z\}$$
$$= \rho_0\omega_0(1-rz) \quad \text{say.}$$

Here, in the expression of $v$, a term $ce^{qz}.x$ is omitted for the sake of avoiding troublesome complications. Though there is some necessity for this term, it is evident that the integration for this term is very easy.

By such simplification we have

$$v_1\rho_1\omega_1 = \rho_0\omega_0 \sum_j \mathrm{A}_j\big[-n_j\sin(m_jx+\theta_j)+\mu_j\cos(m_jx+\theta_j)$$
$$+\{n_j(r-2\mu_j)\sin(m_jx+\theta_j)$$
$$+(\mu_j^2-\mu_jr-n_j^2)\cos(m_jx+\theta_j)\}z\big]e^{\lambda x}.$$

Moreover, if we assume a short part of a stream line near to the ground as a straight line, represented by the equation

$$z = z_1' + \mathrm{D}(x-x_1),$$

it is evident that this value for $z$ can be used for approximate calculation. An example of such straight line, which is used for numerical calculation in this paper, is shown in fig. 3.

Thus we may have

$$\mathrm{W} = \int_{x_1}^{x}(v_1\rho_1\omega_1)\eta e^{-\eta(x-x')}\,dx'$$
$$= \rho_0\omega_0 \sum_j \mathrm{A}_j\eta e^{-\eta x}\int_{x_1}^{x}\{(\mathrm{P}+\mathrm{Q}x')\cos(m_jx'+\theta_j)$$
$$+(\mathrm{P}'+\mathrm{Q}'x')\sin(m_jx'+\theta_j)\}e^{(\eta+\lambda)x'}dx'$$
$$= \rho_0\omega_0 e^{-\eta x}\sum_j \mathrm{A}_j\eta\Big|\{(\mathrm{L}+x'\mathrm{M})\cos(m_jx'+\theta_j)$$
$$+(\mathrm{L}'+x'\mathrm{M}')\sin(m_jx'+\theta_j)\}e^{(\eta+\lambda)x'}\Big|_{x'=x_1}^{x'=x}$$

where . . . (26)

$$\mathrm{P} = \mu_j+(-\mu_jr+\mu_j^2-n_j^2)(z_1'-\mathrm{D}x_1),$$
$$\mathrm{P}' = -n_j+n_j(r-2\mu_j)(z_1'-\mathrm{D}x_1),$$
$$\mathrm{Q} = (-\mu_jr+\mu_j^2-n_j^2)\mathrm{D},$$
$$\mathrm{Q}' = n_j(r-2\mu_j)\mathrm{D},$$

and
$$L = \frac{(P-M)(\lambda_j+\eta)-(P'-M')m_j}{(\lambda_j+\eta)^2+m_j^2},$$
$$L' = \frac{(P-M)m_j+(P'-M')(\lambda_j+\eta)}{(\lambda_j+\eta)^2+m_j^2},$$
$$M = \frac{Q(\lambda_j+\eta)-Q'm_j}{(\lambda_j+\eta)^2+m_j^2},$$
$$M' = \frac{Qm_j+Q'(\lambda_j+\eta)}{(\lambda_j+\eta)^2+m_j^2}.$$

As a numerical example, taking the values of the constants given in Table II., the calculated values of W are shown in Table III. The value of $\eta$ in these is calculated using the

Table III.

| $x$ (km.) ..... | 0 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{W}{N\rho_0\omega_0}$ ..... | 0 | 0·008 | 0·028 | 0·017 | 0·010 | 0·022 | 0·013 | 0·008 | 0·005 |

observed values which are shown in Table IV. The value of W between $x=12$ km. and $x=24$ km. and between $x=36$ km. and $x=48$ km. can be assumed as a mere function proportional to $e^{-\eta(x-x_1)}$, and we can determine the value of $\eta$ by the relation

$$\eta = \frac{\log(\text{ratio of orographic precipitation})}{\text{distance}}.$$

By such method we have values $\eta=0{\cdot}076$/km. between $x=12$ km. and $x=24$ km., and $\eta=0{\cdot}092$/km. between $x=36$ km. and $x=48$ km. The mean of these is $0{\cdot}084$/km. In the present calculation this mean value is used.

The points giving maximum and minimum heights are approximately

| Minimum height. | Maximum height. |
|---|---|
| $x=0$ km. | $x=14{\cdot}3$ km. |
| $x=23{\cdot}4$ „ | $x=34{\cdot}7$ „ |

For the calculation between $x=0$ and the second minimum point the formula (25) is used, assuming $z_1=0{\cdot}5$ km. By this assumption $z_1=0{\cdot}5$ km., a little part of the ground becomes higher than this height. But since such part is short, the same formula is assumed to hold good over this range. For the range between the second minimum point and the second maximum point the formula (26) is used, taking values $z_1'=0{\cdot}4$ km. and $D=0{\cdot}0417$.

The curve representing these values is shown in fig. 3. We can observe that this curve resembles very well the distribution of snow in the mountain region. To compare such values calculated by the formulæ with the actual amount of the precipitation, the amount of snow precipitated on the 20th and the 21st Jan., 1921, is estimated from observations made at several local stations in the vicinity of the region taken in the present example. This precipitation was due to a cyclone which passed through the northern part of Hondo Island from west to east. But the direction of the wind in this locality was N.W. during the time of precipitation. The observed values were plotted on a map, and, drawing the locus of the points of equal precipitation, the amount of the precipitation at the points in the present example was estimated from such locus. Such estimated values are shown in Table IV. The numbers in the row specified as orographic precipitation are the difference between the precipitation at that point and that at $x=0$. These values are also shown graphically in fig. 3.

Table IV.

| $x$, in km. | 0 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 |
|---|---|---|---|---|---|---|---|---|---|
| Precipitation, in mm. | 15 | 30 | 40 | 25 | 25 | 30 | 30 | 25 | 20 |
| Orographic Precipitation, in mm. | 0 | 15 | 25 | 10 | 10 | 15 | 15 | 10 | 5 |

We can see that the part of precipitation which is considered as orographic precipitation has fairly good coincidence with the calculated value.

The absolute value must also be calculable from the formula. According to observations at Yamagata meteorological station *, where observations are made every two hours, the total precipitation during January 1921 was 78·2 mm. on 26 occasions, the total duration being 316 hours. The author has proved in another paper † that the mean duration of such phenomena, calculated from the result of the periodic observation, is longer than the true mean value by one period of observation. In this case, since the period of the observation is two hours, the total duration as computed must be considered as longer by $2\times26$ hours than the true duration. Or the true duration will be $(316-52)\text{h.}=264$ h. Thus the mean precipitation in unit time is

$$\frac{78\cdot2}{264\times3600}\ \text{mm./sec.} = 0\cdot82\times10^{-4}\ \text{mm./sec.}$$

* Yamagata station, about 65 km. distant from the origin of present example.

† Ono, *loc. cit.* Journ. of Met. Soc. of Japan, 1923, p. 127.

On the other hand, at the point $x=30$ km. in the former example the amount of the mean precipitation is estimated to be from 5 times to 10 times greater than that at Yamagata. If we assume the value of W at the point $x=30$ km. to be

$$5\times 0{\cdot}82\times 10^{-4}\ \text{mm./sec.} = 4{\cdot}2\times 10^{-5}\ \text{cm./sec.},$$

we must have

$$4{\cdot}2\times 10^{-5}\ \text{gm./cm.}^2\,\text{sec.} = Nq\omega_0\rho_0\times 0{\cdot}022\ \text{gm./cm.}^2\,\text{sec.}$$

In this equation, if we put

$$\omega_0 = 3{\cdot}745\times 10^{-3}\ (\text{at } 0^\circ\text{C.}),$$
$$\rho_0 = 1{\cdot}293\ \text{gm./cm.}^3,$$

then

$$Nq = \frac{4{\cdot}2\times 10^{-5}}{3{\cdot}745\times 1{\cdot}293\times 10^{-6}\times 0{\cdot}022}\ \text{cm./sec.} = 3{\cdot}9\ \text{m./sec.}$$

And since,

$$\left(\frac{a}{Nq}+1\right)\frac{1}{q} = -0{\cdot}60,$$

$$a = (1-0{\cdot}60\,q)Nq \fallingdotseq Nq = 3{\cdot}9\ \text{m./sec.}$$

This shows that such amount of precipitation in this locality can be produced when $Nq=3{\cdot}9$ m./sec., which is approximately proportional to $a$ (the horizontal wind velocity in the flat region). The locality of the present example is one of the most snowy places in Japan. By this calculation it is shown that an amount of snow in this locality can be produced merely as the result of the orographic effect, if there is a wind of 3·9 m./sec., in the mean, during 264 hours in the month of January. Or, since the orographic precipitation is proportional to the value of $Nq$, if the total mean velocity of wind in January is greater than 1·4 m./sec., such an amount of snow can be precipitated in this locality. With this result we can conclude that the snowy climate in the northern localities of Japan is due chiefly to the orographic precipitation.

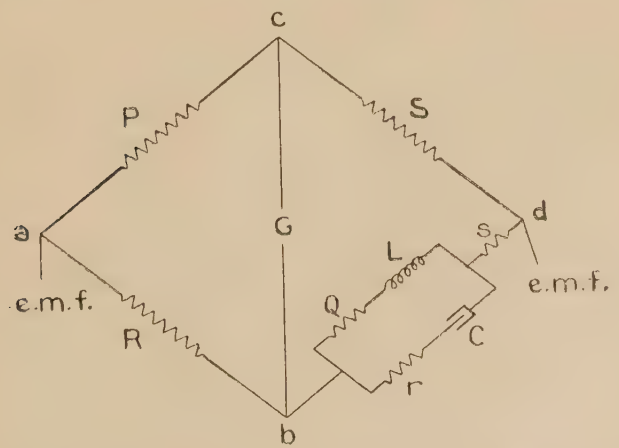
I wish to express my thanks to Prof. T. Okada, Prof. S. Sano, and Prof. T. Terada for their kind advice and encouragement, and to Messrs. K. Morita, K. Funayama, and S. Sekine for their kindness in having sent me the result of their valuable observations. I am also much obliged to Sir Napier Shaw, who has shown much interest in my work and has kindly introduced my paper in England.

XVII. *On a Method of comparing Inductance and Capacity.* *By* Prof. J. M. Ganguli, *M.Sc., Physics Department, Victoria College, Gwalior, India* *.

A FEW years back Prof. W. E. Forsythe †, of the University of Wisconsin, U.S.A., gave an interesting and quite a convenient method for comparing "inductance" and "capacity." It was apparently meant to get rid of some of the difficulties in the ordinary bridge methods, and being tried in his laboratory is said to have given good results. In the ordinary bridge methods there being generally two conditions of balance, the one due to steady current and the other due to reactance (the latter being more or less dependent on the first), it is, as the author says, "often difficult to know just what resistance to choose." To quote from him again, "It often happens that with the capacity and inductance to be compared, the first will be so chosen that the second balance cannot be obtained."

In his method he has tried to get over this difficulty by making the two conditions independent. Finding the experiment interesting I tried it under different conditions with different inductances, and observed that though it was more or less satisfactory in some cases it did not work well, especially when the ohmic resistance of the inductance was increased.

The theory, to quote from the author, is this :—




The connexions are shown in the above sketch. P, R, S, Q, $r$, $s$, are non-inductive resistances, L the inductive coil,

* Communicated by the Author.

† The Physical Review, June 1913.

and C the capacity. $a$ and $d$ are connected to an alternating e.m.f. of frequency $n$.

"To obtain the conditions of balance, make $P=R$, then the impedance of the arm $b$–$d$ must be equal to that of $c$–$d$. Let Z equal the impedance of the part of the arm $b$–$d$ made up of inductance and capacity. We then have

$$\frac{1}{Z}=\frac{1}{Q-iX}+\frac{1}{r+iK}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $X=\omega L$, $K=\frac{1}{\omega C}$, $\omega=2\pi n$, $n$ being the frequency of alternations.

As $P=R$, $Z+s=S$, or

$$\frac{1}{Z}=\frac{1}{S-s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2),

$$\frac{1}{Q-iK}+\frac{1}{r+iK}=\frac{1}{S-s},$$

or clearing

$$(S-s)(r+iK)+(S-s)(Q-iX)=(Q-iK)(r+iX). \quad . \quad (3)$$

Equating the reals

$$(S-s)r+(S-s)Q=Qr+XK, \quad . \quad . \quad . \quad (4)$$

whence

$$XK=(S-s)Q+(S-s)r-Qr. \quad . \quad . \quad . \quad (5)$$

From the imaginaries,

$$(S-s)K-(S-s)X=QK-rX,$$

or

$$K[(S-s)-Q]=X[(S-s)-r]. \quad . \quad . \quad . \quad (6)$$

Now it is known that for the bridge to balance the steady current condition must hold, that is, the resistance of the arm $c$–$d$ must equal that of $b$–$d$, i. e.,

$$S=s+Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From (6) and (7)

$$(S-s)=r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (5) and (8)

$$XK=r^2$$

or

$$L=Cr^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Thus there are three conditions of balance, as follows:

$$S = s + Q, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I.}$$

$$S = s + r, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

$$L = Cr^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{III.}$$

From I. and II. it is seen that Q must equal $r$. Thus to balance the bridge set $Q = r$, and then with direct current on the bridge, obtain the steady current balance $Q + s = S$. Then with the alternating E.M.F. applied to the bridge, obtain the inductive balance with either an A–C galvanometer or a telephone."

In trying the experiment I used different capacities and different inductances. The results were at first more or less satisfactory, though there was usually some uncertainty regarding the exact value of Q when obtaining the balance with the telephone. Changing Q by a few ohms one way or the other had often no appreciable effect on the telephone. In such cases the two limits of Q which the telephone could detect were noted and the mean taken as the value of Q.

The significant thing observed in the experiment was the effect of the ohmic resistance of the inductance. When the ohmic resistance was low and the inductance high in comparison more or less good values were obtained, but the inconsistencies become more and more evident as the resistance of the inductance was increased. In the theory given above in obtaining equation (7) for steady current the resistance of L was ignored, for which purpose Q must be very great. As the resistance of L is increased Q also therefore must be increased very much, so that the resistance of L might still remain insignificant in comparison with $Q + s$. In my experiment Q was also therefore increased, but it was found that in that case the sensibility of the balance was sacrificed so much that sometimes when Q was about 2000 ohms it could be altered through 50 or 60 ohms and sometimes more without being quite perceptible on the telephone.

The method is therefore more of a qualitative than of a quantitative value, and the defects noted above much discount the advantages enumerated by the author.

More detailed investigations, theoretical and experimental, are in progress which will be communicated later on.

XVIII. *On the Dependence of the* K$\alpha$ *Doublet with different Chemical Compounds.* *By* B. B. Ray, *D.Sc., Lecturer, Calcutta University* *.

IT has been found by Lindh † that the absorption edges of elements of lower atomic numbers (P, S, and Cl) are not to the same degree as the lines of the X-ray spectra independent of the physical and chemical conditions under which the element in question exists, but that these edges rather depend on such factors as the valency, that is, whether the element in question is monovalent or divalent, etc., in the chemical compound used as anticathode. Recently Lindh ‡, Tandberg §, and Coster ‖ have extended this work to the K and the L absorption edges of other elements. In this connexion Prof. Bohr ¶ writes : "In agreement with what has been brought to light by investigations on absorption of X-rays in elements of lower atomic number, such as have been performed in recent years in the Physical Laboratory of Lund, we understand immediately that the position and eventual structure of the absorption edges will to a certain degree depend upon the physical and chemical conditions under which the element investigated exists, while such a dependence does not appear in the characteristic emission lines" In the measurement of the K$\alpha$ doublet difference of elements of lower atomic number by Prof. Siegbahn and the author **, it is found that this difference does not vary from element to element in a quite regular manner in the lower elements, but that there are some sudden changes in the iron group and also in the chlorine group, as may be anticipated from the Bohr model. In this experiment pure metals were used as anticathode so far as they were available, and in the absence of pure metals monovalent salts were used. In this connexion Prof. Siegbahn suggested that there may be a difference in $\Delta\lambda$ of the K$\alpha$ doublet when different salts, having different valencies for the element in question, are used as anticathodes. The following experiments were undertaken according to his

* Communicated by Prof. Manne Siegbahn.

† *Diss. Lund*, 1923, also Siegbahn's book, pp. 137–143.

‡ *Ark. f. Mat. Astr. o. Fys.* Bd. xviii. no. 14.

§ *Loc. cit.*

‖ *Zeit. f. Phys.* July 1914.

¶ 'The Theory of Spectra and Atomic Constitution' (Camb. U. Press), p. 120.

** *Ark. f. Mat. Astr. o. Fys.* xviii. no. 19.

suggestion to find out whether there is any difference in $\Delta\lambda$ for four elements (Cl, K, V, and Fe) under different chemical conditions.

### *Experiment.*

A Hadding tube was connected to a vacuum spectrograph, the width of the slit being 0·09 mm., and the photographic plate was placed at a distance of 12·15 cm. from the axis of the reflecting face of the calcite crystal. A current of 10 m.a. at about 50 k.v. was passed through the tube from an air-cooled transformer. In order to get sharp images of the $K\alpha_1$ and the $K\alpha_2$ lines, the time for exposure was necessarily shortened, and in these experiments it only took 10 minutes for all the plates to secure uniform blackness. To put the powdered salt on the copper anticathode, the latter was scratched with a knife and the salt was pressed into the scatches. The anticathode was thoroughly cleaned with a fine iron file when salts were changed, and it was scratched again. This process was repeated whenever new salts were to be put on the copper anticathode. In this experiment only four elements, Cl (17), K (19), V (23), and Fe (26), each having different chemical compositions, were examined.

The following table gives the values of $\Delta\lambda$ for the K$\alpha$ doublet in X-units :—

*Chlorine.*

| Compound. | $\Delta\lambda$ in X.E. |
|---|---|
| KCl | 3·15 |
| $BaCl_2$ | 3·15 |
| $KClO_3$ | 2·99 |
| $Ba(ClO_3)_2$ | 2·98 |
| $KClO_4$ | 2·92 |

*Potassium.*

| Compound. | $\Delta\lambda$ in X.U. |
|---|---|
| KCl | 3·40 |
| $KClO_3$ | 3·22 |
| $KClO_4$ | 3·16 |

*Vanadium.*

| | |
|---|---|
| Va Powder | 3·68 |
| Am. vanadate | 3·60 |

*Iron.*

| Compound. | $\Delta\lambda$ in X.U. |
|---|---|
| Fe (powder) | 4·20 |
| $(NH_4)_2\,Fe(SO_4)_2$ | 4·13 |
| $NH_4\,Fe(SO_4)_2$ | 4·01 |
| $K_4\,Fe(Cy)_6$ | 4·06 |
| $K_3\,Fe(Cy)_6$ | 3·99 |

In this connexion, it may be pointed out that we do not know whether the salts used as anticathode retain their original nature under heavy electronic bombardment. In the above table Cl (17) is monovalent in KCl and $BaCl_2$ and has five valencies in $KClO_3$ and $Ba(ClO_3)_2 + 2H_2O$, and has seven valencies in $KClO_4$ (see Siegbahn's book, p. 142).

So also K (19) is monovalent in KCl and has five valencies in $KClO_3$ and seven valencies in $KClO_4$. The distance between the two K$\alpha$ lines is measured by a comparator from the photographic plate : only the mean value is tabulated above, the error being less than 1 per cent. It is to be noticed that the value of $\Delta\lambda$ decreases as the element investigated attains higher valencies in different chemical compounds.

*Changes of* $\alpha_1$ *and* $\alpha_2$.

The photographic plate holder was then mounted with a fine cross-wire, and in taking the exposure with chlorine and potassium salts it was kept fixed in the required position. In different exposures with different chemical salts, the position of the plate holder was not changed, so that the image of the cross-wire on the photographic plate was formed always at the same angle with KCl, $BaCl_2$, $KClO_3$, $Ba(ClO_3)_2+2H_2O$, $KClO_4$ for chlorine, and with KCl, $KClO_3$, and $KClO_4$ for potassium. The time of exposure in all these plates was only 15 minutes.

The distance between the position of the cross-wire and the corresponding positions of the two lines (K$\alpha_1$ and K$\alpha_2$) was measured by a comparator, and found different for compounds having different valency for the element in question. This shows that the characteristic emission lines are not quite independent of the chemical conditions under which the element investigated exists. It is very difficult to get the exact value of the changes of the K$\alpha_1$ and the K$\alpha_2$ lines, but one can be sure of such changes. In order to find out an accurate value for such a change, a larger number of plates should be examined with many more compounds as anticathodes and possibly with different crystals. The author wishes to examine this point more fully in a future paper.

It gives me great pleasure to thank Prof. Siegbahn for his many valuable suggestions and his keen interest during the progress of the paper. My thanks are also due to Mr. R. Thoraeus for occasional friendly help.

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Sweden.

## XIX. *On a Dynamical Illustration of the Pressure of Radiation.* *By* E. L. Nicolai *.

IN a recent paper Prof. T. H. Havelock † has drawn attention to a dynamical illustration of the pressure of radiation, originally devised by Lord Rayleigh ‡ and Sir Joseph Larmor §. One end of a uniform string is fixed, and the string is supposed to pass through a hole in a moving plate. The pressure of radiation on a moving perfect reflector is illustrated by the pressure of the vibrating string on the moving plate. Prof. Havelock finds the value of this pressure by calculating the rate of decrease of energy of the vibrating string ; he also points out that the pressure so calculated is less than the resolved force on the plate due to the tension of the string.

What is the reason of the pressure on the plate not being equal to the resolved statical tension of the string? The energy-method gives no answer to this question ; in fact the mechanism of the pressure of the vibrating string on the plate remains concealed. In the present note another mode of treatment of the same problem is proposed, which seems to give a clearer idea of the action of the string on the plate. The pressure of the string on the plate can be calculated by the equation of momentum applied to an element of the string which slips through the hole in the moving plate during an infinitesimal interval of time ‖. It is found that in this mode of treatment second-order terms must be retained ; fortunately the transverse displacements of the string are to be calculated to terms of the first order only, so that the usual differential equation of the transverse vibrations of the string remains valid. This method leads to a result in which the well-known formula of Prof. Larmor for the pressure of vibrations on an ideal reflector is included. The result may be slightly generalized by assuming that the plate, while travelling along the string, performs small oscillations in a direction normal to the string. Thus an

* Communicated by the Author.

† T. H. Havelock, "Some Dynamical Illustrations of the Pressure of Radiation and of Adiabatic Invariance," Phil. Mag. xlvii. p. 754 (1924).

‡ Rayleigh, "On the Pressure of Vibrations," Phil. Mag. iii. p. 338 (1902).

§ Sir Joseph Larmor, "On the Dynamics of Radiation," Proceedings of the V. Internat. Congress of Mathematicians (Cambridge, 1913), vol. i. p. 197 (with an Appendix by Prof. T. Levi-Cività).

‖ A calculation of the pressure of vibrations on this line of argument was given by the author in a note, "On the Pressure of Vibrations," Annals of the Petersb. Polytechn. Institute, xviii. p. 49 (1912) [Russian

illustration of the reaction of radiation on a moving source of light is realized.

In the paper quoted Prof. Havelock has given a particular solution of the problem of the vibrating string for the case of a fixed and a moving node, the velocity of the moving node being constant. This particular solution corresponds to a normal vibration of the string in the limiting case of infinitesimal velocity of the moving node. It may be shown that the general solution of the problem is made up by a series of particular solutions of the type indicated *.

§ 1. A uniform endless string is stretched along the axis of $x$; $T_0$ is its statical tension. The string is supposed to pass through a small hole in a movable plate. The string is performing small transverse vibrations, and the plate travels along the string with (uniform or non-uniform) velocity $v$, while performing small oscillations perpendicular to the string. The resolved pressure of the string on the plate in the direction of the $x$-axis is to be calculated.

Let $y$ be the transverse displacement of a point of the string; it is a small quantity of the first order. We shall have to consider longitudinal displacements of the string too; let $u$ be the longitudinal displacement; the string performing transverse vibrations, $u$ is a small quantity of the second order. Let $\xi$ and $\eta$ be the coordinates of the hole in the plate; they are given functions of the time:

$$\xi = f(t), \quad \eta = \phi(t),$$

$\eta$ being a small quantity of the first order.

The extension $\sigma$ of the vibrating string at every point of it is determined to terms of second order by the formula

$$\sigma = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2.$$

The dynamical tension T of the string is

$$T = T_0 + E\omega\sigma.$$

(E Young's modulus, $\omega$ section-area of the string). We suppose the statical tension $T_0$ to be great, so that $\frac{E\omega\sigma}{T_0}$ is a small quantity (of the first order).

* This general solution was given by the author in a paper, "On Transverse Vibrations of a portion of a String of uniformly variable Length," Annals of the Petersb. Polytechn. Institute, xxviii. p. 273 (1921) [Russian].

The values of $\frac{\partial y}{\partial x}$ immediately to the left and immediately to the right of the plate will be denoted by $\left(\frac{\partial y}{\partial x}\right)_1$ and $\left(\frac{\partial y}{\partial x}\right)_2$: and similar notations for other quantities (as $y$, $u$, $\frac{\partial u}{\partial t}$, $\sigma$ etc.) will be used.

§ 2. Now let us apply the equation of momentum to an element of the string of length $vdt$, which slips through the hole in the moving plate during an infinitesimal interval of time $dt$. Resolving along the axis of $x$, denoting the line-density of the string by $\rho$, and assuming $v$ to be positive when directed from left to right (in the positive $x$-direction), we find

$$\rho v dt\left[\left(\frac{\partial u}{\partial t}\right)_1-\left(\frac{\partial u}{\partial t}\right)_2\right]=[\mathrm{T}\cos\alpha_2-\mathrm{T}\cos\alpha_1+\mathrm{N}_x]dt,$$

where $\alpha_1$ and $\alpha_2$ are the inclinations of the string to the axis of $x$ immediately to the left and to the right of the plate and $\mathrm{N}_x$ is the resolved reaction of the plate on the string.

To terms of the second order we have

$$\cos\alpha_1=\frac{1+\left(\frac{\partial u}{\partial x}\right)_1}{1+\sigma_1}=1-\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)_1^2,$$

and in the same manner we find

$$\cos\alpha_2=1-\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)_2^2.$$

Now putting $\mathrm{N}_x=-\mathrm{P}_x$, where $\mathrm{P}_x$ is the resolved pressure of the string on the plate, we have

$$\mathrm{P}_x=\tfrac{1}{2}\,\mathrm{T}\left\{\left(\frac{\partial y}{\partial x}\right)_1^2-\left(\frac{\partial y}{\partial x}\right)_2^2\right\}-\rho v\left\{\left(\frac{\partial u}{\partial t}\right)_1-\left(\frac{\partial u}{\partial t}\right)_2\right\}.$$

This result shows that $\mathrm{P}_x$ is a small quantity of the second order. In calculating $\mathrm{P}_x$ to terms of the second order only, we may put $\mathrm{T}=\mathrm{T}_0$ and so obtain

$$\mathrm{P}_x=\tfrac{1}{2}\,\mathrm{T}_0\left\{\left(\frac{\partial y}{\partial x}\right)_1^2-\left(\frac{\partial y}{\partial x}\right)_2^2\right\}-\rho v\left\{\left(\frac{\partial u}{\partial t}\right)_1-\left(\frac{\partial u}{\partial t}\right)_2\right\}. \quad (1)$$

It is to be noticed, that while calculating $\mathrm{P}_x$ to terms of the second order we have to calculate $y$ to terms of the

first order only. The usual differential equation of transverse vibrations of the string being valid, we have

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

with $$c = \sqrt{\frac{T_0}{\rho}}.$$

Now the condition at the plate ($x = \xi$) is $u_1 = u_2$. Differentiating with regard to time, we find

$$\left(\frac{\partial u}{\partial t}\right)_1 + \left(\frac{\partial u}{\partial x}\right)_1 \cdot \frac{d\xi}{dt} = \left(\frac{\partial u}{\partial t}\right)_2 + \left(\frac{\partial u}{\partial x}\right)_2 \cdot \frac{d\xi}{dt}.$$

Now we have

$$\left(\frac{\partial u}{\partial x}\right)_1 = \sigma_1 - \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)_1^2, \quad \left(\frac{\partial u}{\partial x}\right)_2 = \sigma_2 - \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)_2^2.$$

Assuming the hole in the plate to be perfectly smooth and accordingly $\sigma_1 = \sigma_2$ and putting $\frac{d\xi}{dt} = v$, we obtain

$$\left(\frac{\partial u}{\partial t}\right)_1 - \left(\frac{\partial u}{\partial t}\right)_2 = \tfrac{1}{2} v \left\{ \left(\frac{\partial y}{\partial x}\right)_1^2 - \left(\frac{\partial y}{\partial x}\right)_2^2 \right\}.$$

By (1) we find

$$P_x = \tfrac{1}{2}\rho(c^2 - v^2) \left\{ \left(\frac{\partial y}{\partial x}\right)_1^2 - \left(\frac{\partial y}{\partial x}\right)_2^2 \right\}. \quad . \quad . \quad (2)$$

This result includes the formula of Sir J. Larmor for the case of the ideal reflector; it can also be applied to the case of a travelling source of vibrations. It may be noticed that formula (2) holds good for the case $v^2 < c^2$ only, the fundamental assumptions as to the smallness of $y$, $u$, $\frac{\partial y}{\partial x}$, etc. being valid only for this case.

§ 3. In Prof. Larmor's case we have $\eta = 0$ and $\left(\frac{\partial y}{\partial x}\right)_2 = 0$, the string being at rest to the right of the plate. We obtain

$$P_x = \tfrac{1}{2}\rho(c^2 - v^2)\left(\frac{\partial y}{\partial x}\right)_1^2. \quad . \quad . \quad . \quad . \quad (3)$$

From this result Prof. Larmor's formula is found at once, as is shown in Prof. Havelock's paper. The pressure

$P_x$, given by (3), is less than the resolved statical tension of the string, which is

$$\tfrac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)_1^2=\tfrac{1}{2}\rho c^2\left(\frac{\partial y}{\partial x}\right)_1^2.$$

We see now the reason for this fact. Every element of the string, while slipping through the hole in the plate, is gaining momentum in the direction of $x$. This gain of momentum is due to one part of the resolved statical tension of the string; the other part of the tension produces pressure on the plate.

§ 4. To have a dynamical illustration of a travelling source of light, assume

$$y=F(ct+x) \qquad (-\infty<x<\xi)$$
$$y=\Phi(ct-x) \qquad (\xi<x<\infty).$$

The condition at the plate is $y=\eta$ for $x=\xi$. Hence

$$F(ct+\xi)=\phi(t), \quad \Phi(ct-\xi)=\phi(t), \qquad . \quad . \quad (4)$$

and thus the waves F and $\Phi$, emitted by the vibrating plate, are determined.

Now we have

$$\left(\frac{\partial y}{\partial x}\right)_1=F'(ct+\xi), \quad \left(\frac{\partial y}{\partial x}\right)_2=-\Phi'(ct-\xi),$$

and from (4) we obtain

$$(c+v)F'(ct+\xi)=\phi'(t), \quad (c-v)\Phi'(ct-\xi)=\phi'(t).$$

Hence

$$\left(\frac{\partial y}{\partial x}\right)_1=\frac{1}{c+v}\frac{d\eta}{dt}, \quad \left(\frac{\partial y}{\partial x}\right)_2=-\frac{1}{c-v}\frac{d\eta}{dt}.$$

Substituting in (2), we obtain

$$P_x=\tfrac{1}{2}\rho(c^2-v^2)\left\{\frac{1}{(c+v)^2}-\frac{1}{(c-v)^2}\right\}\left(\frac{d\eta}{dt}\right)^2=-\frac{2\rho cv}{c^2-v^2}\left(\frac{d\eta}{dt}\right)^2.$$

Let $e$ be the energy radiated by the plate in unit time. We have

$$e=\rho c^2\left(\frac{\partial y}{\partial x}\right)_1^2(c+v)+\rho c^2\left(\frac{\partial y}{\partial x}\right)_2^2(c-v)=\frac{2\rho c^3}{c^2-v^2}\left(\frac{d\eta}{dt}\right)^2.$$

Hence

$$P_x=-\frac{v}{c^2}e,$$

and this is the formula of H. A. Lorentz * and M. Abraham † for the reaction of radiation on a travelling source of light.

§ 5. It may be noticed that the same line of argument, by which formula (2) for the resolved pressure $P_x$ of the vibrating string on the plate in the direction of the $x$-axis was found, can also be used for a calculation of the resolved pressure $P_y$ on the plate in the direction perpendicular to the axis of $x$. Using the equation of momentum, we find

$$P_y = -\rho(c^2 - v^2)\left\{\left(\frac{\partial y}{\partial x}\right)_1 - \left(\frac{\partial y}{\partial x}\right)_2\right\}.$$

Applying to the case examined in § 4, we obtain

$$P_y = -2\rho c\frac{d\eta}{dt}.$$

This result accounts for the damping effect of radiation on a source of vibrations.

§ 6. In Prof. Havelock's paper the motion of a string of finite length, one end being fixed and the other passing through a uniformly moving plate, is discussed, and a particular solution of the problem is given. We will show that the general solution of the problem may readily be obtained as a series of terms of the form indicated by Prof. Havelock.

Let $x=0$ be the fixed node and $x=vt$ the moving node ($v=$const.). We have the differential equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

with the boundary conditions: $y=0$ for $x=0$ and $y=0$ for $x=vt$.

The general solution is

$$y = F(ct+x) - F(ct-x), \quad . \quad . \quad . \quad . \quad (5)$$

where F satisfies the functional equation

$$F\left(\frac{c+v}{c-v}z\right) = F(z).$$

Now put $z=e^w$, $w=\log z$. Denoting $F(e^w)=\psi(w)$, we have

$$\psi\left(w+\log\frac{c+v}{c-v}\right) = \psi(w).$$

* H. A. Lorentz, *Enzykl. der mathem. Wissensch.* Bd. v. art. 14, p. 270.
† M. Abraham, 'Theorie der Elektrizität,' Bd. ii. p. 119.

Thus $\psi(w)$ is a periodic function, the period being $\log\frac{c+v}{c-v}$. Expanding in a Fourier series and denoting

$$\frac{2\pi}{\log\frac{c+v}{c-v}}=\alpha,$$

we obtain

$$\psi(w)=\sum_{n=0}^{n=\infty}\{C_n\sin n\alpha w+D_n\cos n\alpha w\},$$

$C_n$ and $D_n$ being arbitrary constants. Hence

$$F(z)=\sum_{n=0}^{n=\infty}\{C_n\sin(n\alpha\log z)+D_n\cos(n\alpha\log z)\}.$$

Substituting in (5), we obtain after some reductions the general solution of our problem in the form

$$y=\sum_{n=1}^{n=\infty}\sin\left[\frac{n\alpha}{2}\log\frac{ct+x}{ct-x}\right]\left\{A_n\cos\left[\frac{n\alpha}{2}\log\left(\frac{t^2}{t_0^2}-\frac{x^2}{c^2t_0^2}\right)\right.\right.$$
$$\left.\left.+B_n\sin\left[\frac{n\alpha}{2}\log\left(\frac{t^2}{t_0^2}-\frac{x^2}{c^2t_0^2}\right)\right]\right\},\quad\ldots\quad(6)$$

$t_0$ being the initial time (and hence $ct_0$ the initial length of the string) and $A_n$ and $B_n$ denoting arbitrary constants, related in a simple manner to the constants $C_n$ and $D_n$.

The constants $A_n$ and $B_n$ are defined by the initial conditions of the problem. Let $y=f(x)$, $\frac{\partial y}{\partial t}=g(x)$ for $t=t_0$ and $0<x<ct_0$. By (5) we easily find

$$F(ct_0+x)=\tfrac{1}{2}f(x)+\frac{1}{2c}\int_0^x g(x)dx,\quad(-vt_0<x<vt_0),$$

where $f(-x)=-f(x)$ and $g(-x)=-g(x)$ is assumed. Also

$$\left.\begin{aligned}A_n&=\frac{2\alpha}{\pi}\int_{-vt_0}^{vt_0}F(ct_0+x)\sin\left\{n\alpha\log\left(1+\frac{x}{ct_0}\right)\right\}\frac{dx}{ct_0+x}\\B_n&=-\frac{2\alpha}{\pi}\int_{-vt_0}^{vt_0}F(ct_0+x)\cos\left\{n\alpha\log\left(1+\frac{x}{ct_0}\right)\right\}\frac{dx}{ct_0+x}\end{aligned}\right\}.\quad(7)$$

It is easily verified that the limiting values of (6) and (7) for infinitesimal velocity $v$ are the well-known expressions relating to the vibrations of a string of constant length.

Polytechnic Institute, Leningrad,
Sept. 29, 1924.

XX. *On the Permanent Electret.* *By* MOTOTARÔ EGUCHI, *Professor of Physics, Higher Naval College, Tokyo, Japan* *.

*Introduction.*

"ELECTRET" is the name given to the dielectric which is electrized † permanently by a special treatment due to the author. Some waxes and resinous materials have moderate conductivity in their liquid state, while they are very good insulators in the solid state. The electrical conductivity of these materials varies gradually with the degree of solidification, and when the materials get moderately hard the conductivity becomes practically nil. The author let solidify a mixture of these materials in a strong electric field applied through all the time in which the solidification went on. The dielectric taken out of the field, after having been cooled sufficiently, showed very strong polarities on its two surfaces which were kept in contact with the electrode plates during the preparation. The electrization of such a dielectric could not be destroyed by several treatments, such as touching by Bunsen flame, exposure to X-rays, planing with knife, washing with some solvents, &c. Nor did it die away after the lapse of many years. From several subsequent studies, it has become evident that the electrical change of the dielectric is not of a superficial nature, but that it is a permanent internal change within the material.

The name "Permanent Electret" or simply "Electret" was given to the special dielectric at the ordinary meeting of the Physico-Mathematical Society of Japan, February 21, 1920 ‡. Later, I found in Oliver Heaviside's 'Electrical Papers' the section with the title "Electrization and Electrification. Natural Electret" §. In this paper he proposed for the first time to use the term "Electret" to fill the want for describing an intrinsically electrized body, and some possibilities of electrets were discussed on the theoretical points of view. The present method of preparation was,

* Communicated by the Author.

† The term "Electrization" is used after Heaviside to signify the internal electrical change of a material, which is different from superficial electrification.

‡ Mototarô Eguchi, Phys.-Math. Soc Japan, ser. 3, vol. ii. no. 7 (1920).

§ Oliver Heaviside, 'Electrical Papers,' vol. i. § xii.

however, obtained independently and also in a way utterly different from his discussions.

The electret shows so great an intensity of electrization that the electric force exerted in front of the surface of the electret may attain the greatest sustainable value in the atmosphere. The permanency is also so good that we have detected no sensible decaying for these three or more years since the preparation.

## § 1. *Preparation of the Permanently Electrized Dielectric.*

From the study of the variation of conductivity with the solidification of waxes and some other materials *, I came to think it possible to get a permanently electrified or more correctly electrized dielectric by allowing some kind of waxes to solidify in a strong electric field.

After many trials, it has been ascertained that the disk-shaped electret of a certain size is one of the most convenient forms for several reasons, except for the cases in which special studies are wanted. The method of preparation of the disk-electret is, therefore, briefly described here.

A shallow circular metal basin B (fig. 1), depth 1 cm., diameter 20 cm., is put on a pole-plate $P_1$, a little larger in diameter than the basin. The plate itself is placed on three sulphur insulators $S_1$, $S_2$, $S_3$, which are laid on a wooden tripod stand T capable of levelling by the three screws $f_1, f_2, f_3$.

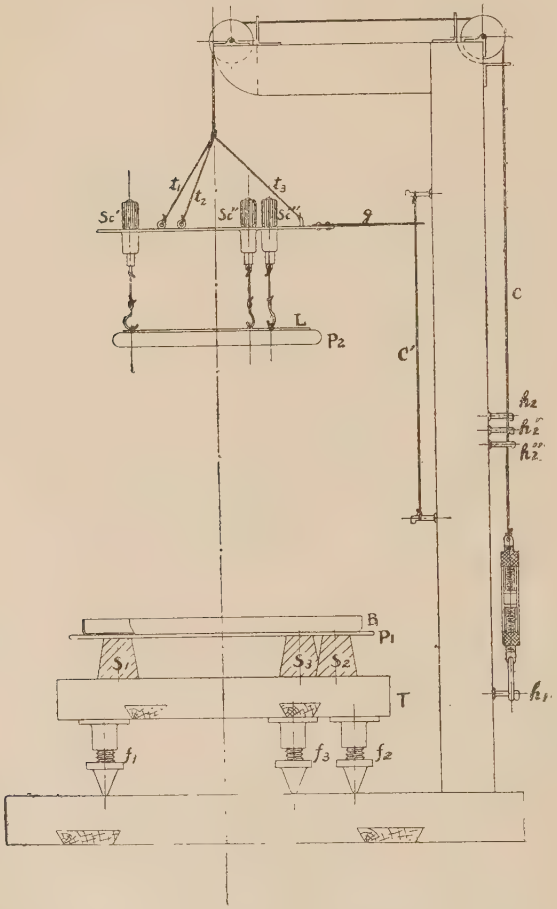
As the other electrode, a hollow metal disk $P_2$ is suspended by three insulating cords, $t_1, t_2, t_3$. These cords are made of Japanese fishing-lines, Tegusu (a kind of thick silk thread), whose surfaces have been treated with a mixture of some waxes and resin to get rid of the surface leakage. The basin and the disk-electrode are coated with tin-foils (not shown in the figure). The lead sheet L is used as a weight to cause the foil tightly to attach itself to the disk.

As the preparatory adjustment, the levelling of the basin B is first effected by three screws $f_1, f_2, f_3$; secondly, the hook at the end of the cord C is transferred from the arrester $h_1$ to another $h_2$, and the plate $P_2$ is so regulated by three screws Sc′, Sc″, Sc‴ that its lower surface comes

* Mototarô Eguchi, Phys.-Math. Soc. Japan, ser. 3, vol. i. nos. 10–11 (1919).

just up to the margin of B. Another cord C′ stretched through a hole perforated near the end of the guider $g$ serves to prevent the oscillation of the electrode $P_2$.

Fig. 1.




After raising the disk-electrode $P_2$ up to a sufficient height, the material melted at a temperature far above its melting-point (about 130° C.) is poured in to fill up the basin B, and then $P_2$ is put on the melted dielectric so that it rests just on the surface of the dielectric. The air bubbles, if any, on the surface of the melted material may

be easily taken off by touching with a small Bunsen flame.

Care was taken to adjust the voltage of the high tension source in order to prevent any undesirable effect such as brush discharges between the electrodes along the surface of the dielectric when it solidified partially. Tin-foils were applied on the electrodes, so that not only did the dielectric not adhere directly to the metal surface, but it might also contract freely as it cooled, and when the dielectric solidified completely the disk-shaped material might be easily taken off from the basin. Moreover, with the disk-electret thus obtained, both surfaces upper (U) and lower (L) might be studied at will.

### § 2. *The Charge of Temporary Nature and the Permanent Electrization of the Electret.*

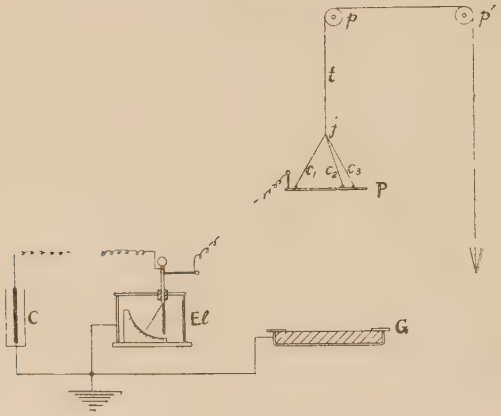
As the material for the permanent electret, a mixture consisting of equal parts of carnaüba-wax and resin with or without a certain amount of bees'-wax was found to be very good as regards the intensity of electrization and the permanency. If we prepare a disk-electret of this material, joining for instance the upper surface (U) to the positive pole of the high tension source, and the lower surface (L) to the negative, we first detect in general a surface charge of negative sign on the upper surface, and one of positive sign on the lower, soon after we take the electret out of the preparing arrangement. But these kinds of charges are of temporary nature, and they die away gradually, in a comparatively short time (in one or two days). After the complete decay of such charges, the surface charges of opposite signs respectively to the above gradually grow up. The positive charge on the upper surface and the negative on the lower, which are respectively of the same signs as those of the applied poles of the source during the preparation, tend to grow to their ultimate values in a few days. The manner of growth of these charges on both surfaces and their values do not vary much even when the method of preparation is modified in some way : for example, when both electrodes are completely insulated from the earth, or when either one of the electrodes is completely earthed and the other insulated. The permanency of these charges is so good that we cannot detect any sensible decay after many years, as will be shown later on (see § 8). We shall call these surface charges of the electret the free charges due to its proper electrization.

### § 3. *On the Method of Measuring the Surface Density of the Free Charge.*

Before entering upon the details of the behaviour of the electret, it will be convenient to describe the method of measuring the surface density of the free charge which was found to be the most simple and reliable.

The arrangement is shown in fig. 2. P is an induction plate of a circular form, diameter 12 cm., with a guard-ring G. The induction plate may be kept at any desired height by means of three insulating suspension cords $c_1$, $c_2$, $c_3$, and a thread $t$ attached to the common joint $j$ of these cords, passing over pulleys $p$ and $p'$. The size of the plate was

Fig. 2.

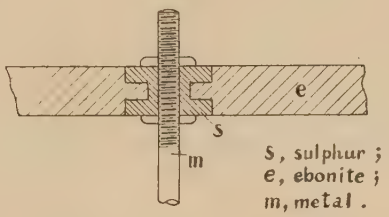



chosen so large that the electrization of the portion of the dielectric under the plate was quite uniform, and the guard-ring G with a small air-gap was employed for eliminating the end effect on the induction plate. To the induction plate, a calibrated electrometer El and a variable condenser C are connected by means of soft springs of thin wires. Putting the disk-electret together with its metal basin in the proper position under the induction plate, the earthed guard-ring is first laid on the surface to be measured. The induction plate is then put in its position on the surface, the system being always kept earthed during this process. Disconnecting the earth line of the plate system, and lifting the plate to such a height that the influence of the dielectric on the plate was no longer sensible, the electrometer reading

was taken. As to the insulation of the condenser, special care was taken to get rid of the surface leakage. The metal rods of the condenser plates were suspended through the central holes of the sulphur pieces moulded in the form shown in fig. 3.

Such a moulded piece of sulphur has a sufficient mechanical strength, and it was ascertained that the condenser might be used in any wet atmosphere without fear of any appreciable leakage.

Fig. 3.




The surface density $\sigma$ of the free charge of the electret may be calculated from the observed numbers as follows :—

$$\sigma = \frac{C \cdot V}{\pi r^2},$$

where C is the capacity of the condenser system, V the potential read by the electrometer, $r$ the effective radius of the circular surface of the induction plate (radius of the plate + one-half of the air-gap). Any conducting agent, such as moisture, dust, free ions, &c., if present on the dielectric surface, may reduce the electrostatic induction on the plate. It is very difficult to keep the surface quite free from such disturbances. Moreover, the irregularity of the surface of the electret may have an influence on the induction of the plate. The value of $\sigma$ estimated as above may, therefore, be somewhat smaller than the true value.

The sign of the induced charge was determined every time and that opposite to this sign was taken as that of the free charge.

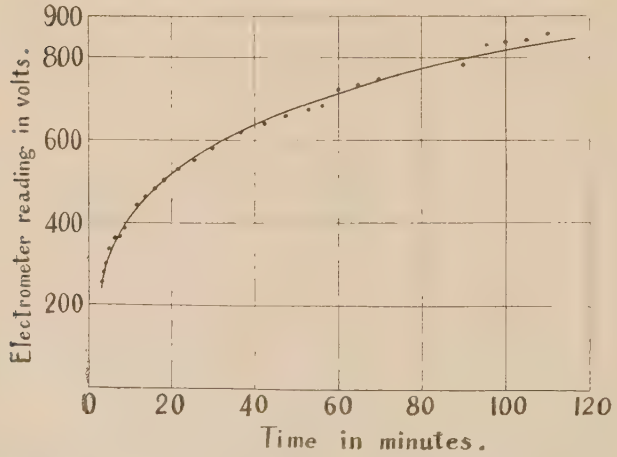
### § 4. *Action of X-rays on the Electret.*

If the surfaces of a disk-shaped electret are exposed to X-rays for some time, the free charges on the surfaces apparently disappear. The surface density of the free charge reappears, however, gradually, and attains its final proper value estimated before the exposure. However many times

the exposures are repeated, the recoveries are always complete. By the time of exposure, the nature of the X-rays and the intensity of the rays, the rate of recovery is influenced in greater or less degree. But in any case, when the free charge has recovered itself completely, the final surface density is always the same, and shows a definite value proper to the electret.

Several methods of exposing the disk-electret to X-rays were tried. In one case, the X-rays were sent in the direction normal to the disk-surface, allowing the rays to pass through the dielectric material. In another case, the X-rays were let go along the disk-surface, the edge of the disk being protected from the rays by a lead band of a sufficient breadth, and thus the rays were cut off and could not go through the dielectric material. In either case, we did not find any material difference in the temporary disappearance of the free charges and their spontaneous recoveries.

Fig. 4.




This temporary disappearance of the free charge and its recovery afterwards might be explained on the view that the ions in air produced by X-rays are caught on to the surface by the charges of opposite sign respectively, and when the action of the X-rays ceases, the ions fly off spontaneously from the surface by their kinetic energy of agitation, opposing the electrostatic attraction of the free charges of the dielectric surfaces on the ions. A few examples of the observed facts are given below.

In fig. 4 the time curve of the recovery of the free charge

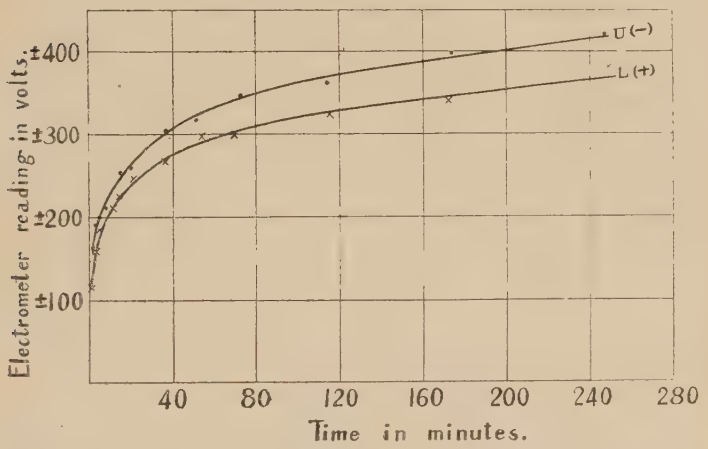
after an exposure to very soft X-rays is plotted, the surface of the disk-electret being kept bare during the recovery. The recovery obeyed very nearly an exponential law,

$$V' - V = (V' - V_0)e^{-\lambda t},$$

where V is the potential of the electrometer at time $t$, and $V'$ the final potential which corresponds to the ultimate free charge.

In fig. 5 the simultaneous recoveries of the charges of both surfaces, upper and lower, of one disk-electret are given.

Fig. 5.




The small difference of the ultimate values of the surface densities for the two surfaces will probably be due to the difference of the condition of the two surfaces, such as the non-uniformity of the electrization near the rim of the surface, &c.

### § 5. *Evidences for Permanent Internal Electrization in the Dielectric.*

Many experimental studies were carried out for testing the author's view that the electrification measurable as the surface density of opposite signs appearing respectively on the two surfaces of the special electret are the free charges of the permanent electrization within the material. The brief description of the results of these studies will be given below.

(1) Temporary disappearance of the free charges by X-ray exposure and their complete recovery after the withdrawal of the rays (as stated in the preceding section).

(2) The behaviour of the free charges after the treatment of the surface by a Bunsen flame is very similar to the effect of X-ray exposure.

When the surface of the electrized dielectric was treated with a Bunsen flame in such a degree that some surface layer of the dielectric material was melted and again solidified after the withdrawal of the flame, the surface charge was very much reduced, or sometimes it completely vanished, if measured soon after the treatment. This disappearance was, however, only temporary, and the free charge recovered itself, as was seen in the case after an X-ray exposure.

(3) Effect of planing with knife.

I planed with a sharp knife both upper and lower surfaces of an electrized dielectric to some depth, so that the old surfaces no longer remained. The free charges were apparently reduced to a small amount, but they recovered gradually as they did after the exposure to X-rays, the only difference from the latter case being that the recovery was somewhat irregular. When the free charges had recovered completely, I tried the knife treatment once more to the depth of some mms., after which the recovery was also similar and perfect, and the ultimate values of the free charges were very near to those before the treatment.

(4) Effect of washing with some reagents.

I studied the effects of washing the electret surfaces with some reagents such as water, acids, alkalis, absolute alcohol, ether, benzene, &c. Though the free charges disappeared after such a treatment, they always recovered some time after the dielectrics were made completely dry. The results were, thus, also affirmative for the permanent internal electrization of the material.

(5) Comparison of two pieces of dielectrics of the same materials, one of which was electrized by the special treatment, the other merely superficially charged by means of an ordinary process.

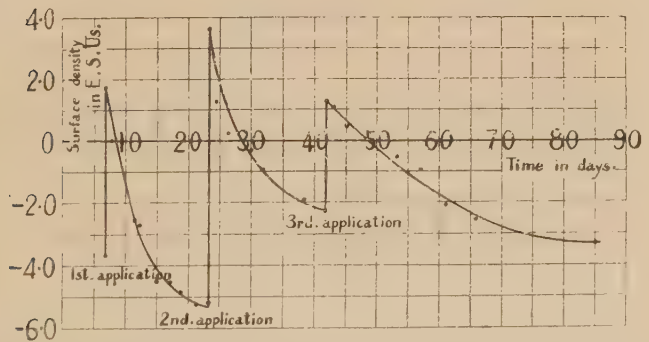
The dielectric material was solidified in a metal basin under conditions otherwise the same as in the preparation of the special electret in every respect, but applying no electric field. After the solidification was completed, the material was put under the action of an electric field for a sufficiently long time of one hour or more. Such a dielectric, soon after the treatment, takes strong surface charges of opposite polarities on its surfaces. Though these surface charges

displayed some weak residual charges, their intensity and durability were of very small order quite incomparable with those of the electret.

(6) Effect of electric force on the electret.

Different electric forces were applied on some disk-electrets, and the effect of such forces on the electrization of the material was observed. I took two of them of which the sense of electrization was known. For convenience' sake, the electric force in the same sense as the field employed in the preparation for an electret will be called "positive," and that of the opposite sense "negative." To the first disk $A_3$ a negative electric force was applied for thirty minutes, and afterwards the variation of the free charges was observed. The results of the observation repeated three times successively are plotted in fig. 6, the surface density of the

Fig. 6.

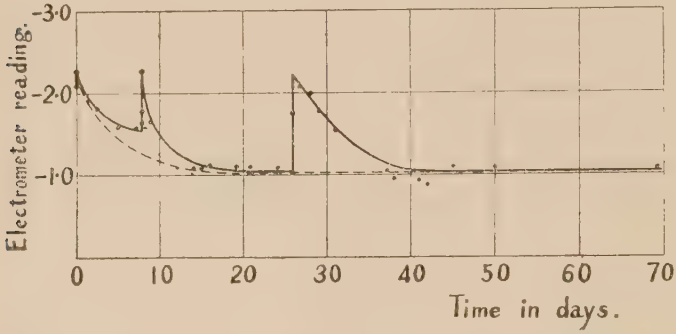



upper surface U only being shown in the figure, since the other surface L showed a behaviour similar to U in every respect but with the sign of its charge opposite.

As we see on the figure, seven days after the preparation, the disk had a free charge of negative sign on surface U and of positive on L, nearly settled at their full values. By the application of the negative electric force for thirty minutes, the surfaces got moderately large surface densities of the signs respectively opposite to those of the free charges of the proper electrization. These temporary charges, however, decayed rapidly, and the gradual growth of the surface charges due to the original electrization followed. When the rate of growth of the surface charge became very small, the second and the third applications of the negative electric forces were tried successively, and the results were always of the same nature as in the first application.

With the second disk $W_2$, the effect of applying the positive electric force was studied. Though small temporary increases of the surface charges were observed after the application of this force, they settled rapidly to their proper value due to the electrization. The results are depicted in fig. 7.

Fig. 7.




From these experiments we may safely conclude that, by the application of an electric force, positive or negative, on a prepared electret, the surface electrification may be disturbed in some degree, but the permanent electrization of the electret remains almost unaffected.

## § 6. *On the Intensity of Electrization of the Permanent Electret.*

From many experimental studies made by the author * the internal electrical change of the permanent electret was concluded to be of the nature of "polarization," *i. e.* every portion from element to element of the dielectric material was in a state of having acquired equal and opposite electrical polarities on its two ends.

The surface density of the free charge measured by means of the method described in § 4 may, therefore, be taken as the intensity of permanent electrization within the dielectric. Most electrets of which the materials consist of carnaüba-wax and resin with or without some percentage of bees'-wax, have so great intensities of electrization that they exceed the greatest possible surface densities of the electrifications charged on some good insulators such as sealing-wax, shellac, sulphur, &c., by means of mechanical friction. The following numbers are given to show the comparison of these surface densities.

* Mototarô Eguchi, Phys.-Math. Soc. Japan, ser. 3, vol. v. no. 8 (1923).

TABLE I.

Maximum attainable surface densities by mechanical friction *.

| Material. | Rubbed on wool. Max. surf. density. | Rubbed on silk. Max. surf. density. |
|---|---|---|
| Sealing-wax ..... | 5·70 E.S.U. | 5·40 E.S.U. |
| Shellac ..... | 5·57 „ | 5·12 „ |
| Sulphur ..... | 5·48 „ | 5·39 „ |
| Ebonite ..... | 4·82 „ | 4·43 „ |
| Amber ..... | 4·18 „ | 4·11 „ |
| Glass ..... | 2·65 „ | 2·90 „ |

TABLE II.

Surface densities of the free charges of the internal electrization of the permanent electrets.

Disk-electret $O_1$, prepared Oct. 27, 1919.

| Date ($d$). | Time since the preparation ($t$). | Surf. ($s$). | Surf. density ($\sigma$). | Electric intensity in air in front of the surf. (E). |
|---|---|---|---|---|
| Mar. 22, 1921... | $1^{y.}\ 5^{mon.}$ | U | −5·7 E.S.U. | −72 E.S.U. |
| | | L | +6·2 „ | +78 „ |
| Mar. 3, 1922... | $2^{y.}\ 5^{mon.}$ | U | −6·5 „ | −82 „ |
| | | L | +6·2 „ | +78 „ |

Disk-electret $R_1$, prepared Oct. 30, 1919.

| $d$. | $t$. | $s$. | $\sigma$. | E. |
|---|---|---|---|---|
| Mar. 22, 1921... | $1^{y.}\ 5^{mon.}$ | U | −5·6 E.S.U. | −70 E.S.U. |
| | | L | +5·6 „ | +70 „ |
| Mar. 3, 1922... | $2^{y.}\ 5^{mon.}$ | U | −6·0 „ | −75 „ |
| | | L | +5·8 „ | +73 „ |

The electric intensities in air in front of the surfaces of the electret are calculated from $E=4\pi\sigma$, and are given in the last column in the above table. Lord Kelvin found that the electric intensity sustainable in air was 130 E.S.U., and therefore the corresponding surface density was 10 E.S.U. These numbers might, however, hardly be realized in a very good condition of air. The numbers of the permanent electrets are not much smaller than these limiting values. It will be observed, in what a higher order of intensity of electrization the electrets are than the ordinary residual electricities in dielectrics hitherto known.

* L. Graetz, *Handbuch der Elektrizität und des Magnetismus*, Bd. i. Saite 10 (1918).

### § 7. *On the Permanency.*

The disk-shaped electrets are kept usually with their surfaces covered with earthed tin-foils. Under this surface condition, the surface density of the free charge of the electrization remains nearly constant for many years, though small fluctuations may occur probably due to the atmospheric conditions such as the temperature and humidity. Some disk-electrets were tested by keeping them in this condition during three years or more since their preparation, their surface densities being measured at moderate intervals. No natural decay of the surface densities was observed. In the following table two examples of the records of the electrets kept in such a condition are given.

TABLE III.

Surface density of the free charge of the electrization.

Disk-electret $O_1$.

| Date..... | Oct. 27, 1919. | Dec. 23, 1920. | Dec. 24, 1921. | Mar. 3, 1922. | Dec. 20, 1922. |
|---|---|---|---|---|---|
| Surf. density in E.S.U. or other remarks. | prepared | U −4·5<br>L +5·6 | U −5·8<br>L +5·1 | U −6·5<br>L +6·2 | U −5·1<br>L +7·2 |

Disk-electret $R_1$.

| Date..... | Oct. 27, 1919. | Dec. 23, 1920. | Dec. 24, 1921. | Mar. 3, 1922. | Dec. 20, 1922. |
|---|---|---|---|---|---|
| Surf. density in E.S.U. or other remarks. | prepared | U −5·1<br>L +5·6 | U −5·8<br>L +4·7 | U −6·0<br>L +5·8 | U −6·2<br>L +7·2 |

If the electret is kept with its surfaces bare the surface density apparently decays in some degree. The final reduced value of the surface density seems, however, to be kept at some amount different from zero. It is worth noticing that, for observing these decays definitely, it is necessary to keep the electret on some pointed supports in a dry and dust-free air in a desiccator. Otherwise a coating of dust attracted by the free charge behaves as an earthed conductor.

If an electret, whose free charge has thus decayed to a small amount, is again kept covered with tin-foils applied on both surfaces the free charge recovers itself gradually, and at last its surface density attains its proper value. The length of time during which the surfaces are kept bare may affect somewhat the rate of recovery of the surface density when the electret is again covered. The final surface

density seems, however, to take the definite value proper to the electret.

The disk-electret $M_2$ was taken for this kind of tests, and the results are given in Table IV.

TABLE IV.

Disk-electret $M_2$.

| Date... | Nov. 30, 1920. | Dec. 10, 1920. | Dec. 15, 1920. | Dec. 18, 1920. | Oct. 18, 1921. |
|---|---|---|---|---|---|
| Surf. density in E.S.U. or other remarks. | prepared | U −5·1<br>L +6·7 | U −4·2<br>L +5·8 | U −2·5<br>L +2·5 | U −0·1<br>L +0·05 |
| Surf. condition. | ← | kept covered with tin-foils. → | | kept bare in ordinary atmosphere; both surfaces were coated with dust. → | ← |

both surfaces cleaned with feather brush and then exposed to X-rays.

| Oct. 22, 1921. | Nov. 12, 1921. | Jan. 14, 1922. | Jan. 30, 1922. | Mar. 6, 1922. |
|---|---|---|---|---|
| U −0·2<br>L +0·1 | U −3·5<br>L +3·1 | U −0·1<br>L +0·08 | U −2·5<br>L +1·9 | U −4·2<br>L +3·2 |
| kept in a desiccator with both surfaces covered. → | ← kept in a desiccator with surfaces bare. → | | ← kept in a desiccator with surfaces covered. → | |

surf. density moderately great; permanency good.

*Summary.*

1. The Permanent Electret was made of a certain mixture of wax and resin, keeping it in a strong electric field during its solidification.

2. The nature of the electrical change of the dielectric is a permanent internal electrization of the material entirely different from the superficial electrification.

3. The electrization cannot be destroyed by different treatments applied on the surface such as Bunsen-flame treatment, X-ray exposure, washing with some solvents, planing with knife, applying electric force in any sense, &c.

4. The intensity of electrization of the electret attains such a high degree that the electric intensity in air in front of the electrized dielectric takes nearly the greatest sustainable value.

5. The permanency of the electrization is also extremely

good, so that we have detected no decaying for three years or more since the preparation.

In conclusion, the author wishes to express his best thanks to Dr. Terada, Prof. of Tokyo Imperial University, and Dr. Honda, Prof. of Tohoku Imperial University, for their continued interest in the present investigation.

Physical Laboratory,
Higher Naval College,
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XXI. *Abnormal Reflexion of X-Rays.* *By* J. H. Smith, *M.Sc., Assistant Lecturer in Physics, University College, London* *.

Clarke and Duane† have found, by the ionization spectrometer method of Bragg, certain diffraction maxima which do not obey the two laws governing the Bragg maxima, namely—the angle of incidence equals the angle of reflexion, and that the orders of spectra are given by

$$n\lambda = 2d \sin \theta,$$

where $\lambda$ is the wave-length of the rays and $d$ the distance between planes concerned.

Prior to this Duane ‡ had discovered maxima obeying the Bragg laws, but due to characteristic radiation produced by fluorescence of atoms in the crystal.

If this set of maxima exists it is easily shown that an "abnormal" set must also exist; and also that this second set will be of greater intensity than the first set. This abnormal set also exists when there is no fluorescence.

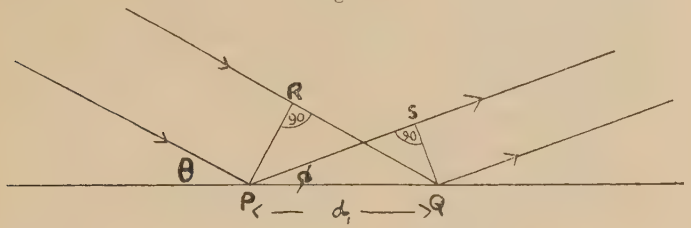
These abnormal maxima are due to the periodicity of scattering centres in the planes themselves, and, unlike the main Bragg maxima, the positions of the abnormal sets will vary as the crystal is rotated about the normals to the planes producing them.

Consider a plane wave of homogeneous radiation incident on a set of planes a distance $d_2$ apart, at a glancing angle $\theta$. Bragg § considers each plane as continuous, and this is sufficient for the set of maxima which is primarily related to $d_2$. For the general maximum the planes of spacing $d_2$ can be considered as made up of scattering centres with a spacing $d_1$ in the plane of incidence.

* Communicated by the Author.
† Proc. Nat. Acad. Sci. ix. 4. p. 131 (1923).
‡ Proc. Nat. Acad. Sci. viii. 5. p. 90 (1922).
§ 'X Rays and Crystal Structure,' 2nd edition, p. 12.

The first condition that there shall be a maximum in any direction making an emergent glancing angle $\phi$ is then :

Fig. 1.



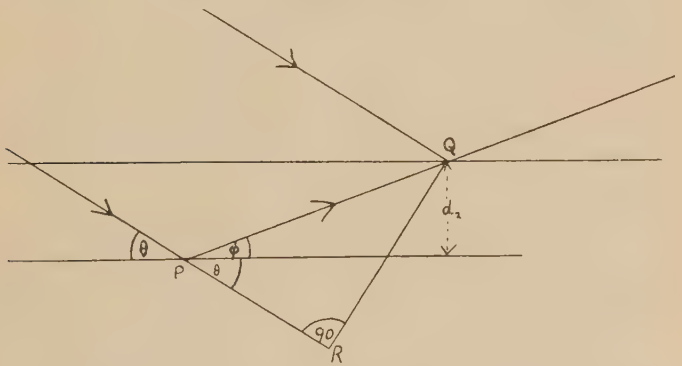

$$PS-RQ=n_1\lambda, \quad \text{where} \quad n_1=0, 1, 2, \text{ etc. ;}$$

$$i.e., \qquad \cos\phi-\cos\theta=n_1\lambda/d_1 ;$$

$n_1=0$ gives the Bragg maxima, and is independent of $d_1$.

Instead of Bragg's second condition, the relation will be

Fig. 2.




$$PQ-PR=n_2\lambda, \quad \text{where} \quad n_2=1, 2, 3, \text{ etc., but not } 0;$$

$$i.e., \qquad \frac{1}{\sin\phi}-\frac{\cos(\theta+\phi)}{\sin\phi}=\frac{n_2\lambda}{d_2},$$

which gives the Bragg maxima when $\theta=\phi$.

If $d_1$ and $d_2$ are known, the values of $\theta$ and $\phi$ corresponding to various values of $n_1$ and $n_2$ can be found.

In order to explain the first set of fluorescent maxima Clarke and Duane assume that there is a definite phase relation between the absorbed wave and the emitted phosphorescent radiation. Their first set of maxima, which obey the Bragg laws, must be due to the interference of this fluorescent radiation with the same radiation scattered from atoms other than its source. The abnormal maxima will be

due to the interference of the initial fluorescent radiation without any secondary scattering, and will thus be of greater intensity than the first Duane set.

The laws obeyed will be similar to those above. If $\lambda_1$ is the wave-length of the incident radiation and $\lambda_2$ that of the fluorescent radiation, the two conditions become

$$\frac{\cos\phi}{\lambda_2} - \frac{\cos\theta}{\lambda_1} = \frac{n_1}{d_1}$$

and

$$\frac{1}{\lambda_2 \sin\phi} - \frac{\cos(\theta+\phi)}{\lambda_1 \sin\phi} = \frac{n_2}{d_2},$$

with $n_1$ and $n_2$ as before.

Since Duane used white radiation it is impossible, from his data, to do more than show that the various values of $\lambda_1$ lie within the limits of his spectral range.

McKeehan * has solved the question of abnormal maxima for a cubic lattice only. The above method is applicable to any system.

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## XXII. *The Mutual Action of Charged Particles in Liquid Media.* *By* Professor E. F. Burton *and* J. E. Currie, *M.A.*†

### I. *Electroendosmose and Related Phenomena.*

THE term electroendosmose is applied to the phenomenon of the transfer of liquids through capillary tubes under the action of an electric field set up throughout the tubes by electrodes placed at the ends of the tubes and maintained at a given difference of potential. The experiments on single capillary tubes have been extended to porous diaphragms which are treated merely as collections of fine capillaries. The laws governing this action have been very fully worked out by a succession of scientists.

Although this phenomenon of electroendosmose has been given particular attention, it is merely one of four phenomena which treat of the electrical forces set up by the relative motion of solids and liquids or of such relative motions set up by applied electromotive forces. The correlation of these phenomena may be seen in Table I.

* Opt. Soc. Amer. and Rev. Sci. Inst. Dec. 1922, p. 989.

† Communicated by the Authors. Read before joint session of Sections A and B, British Association for the Advancement of Science, Toronto meeting.

TABLE I.

| A. Motion caused by applied E.M.F. | B. E.M.F. set up by impressed motion. |
|---|---|
| I. Liquid moving through stationary solid tubes. Electroendosmose (1). | I. Liquid forced through stationary solid tubes. Helmholtz-Billiter (3). |
| II. Solid particles moving through stationary liquids. Cataphoresis (2). | II. Solid particles dropped through stationary columns of liquid. Dorn effect (4). |

(1) Wiedemann, *Elektricität*, i. p. 1007 (1893).
Quincke, *Pogg. Ann.* cxiii. p. 513 (1861).
Helmholtz, *Ann. d. Phys.* vii. p. 337 (1879); Memoirs Lon. Phys. Soc. 1888.
Lamb, Brit. Assn. Rep. p. 495 (1887).
Perrin, *Jour. Chim. Phys.* ii. p. 607 (1904); iii. p. 50 (1905).
Elissafoff, *Zs. f. Phys. Chem.* lxxix. p. 385 (1912).
Briggs, Bennett, and Pierson, J. of Phys. Chem. xxii. p. 256 (1918).
(2) Burton, Physical Prop. of Coll. Sol. (Longmans, Green, 1921, p. 132).
M. E. Laing, J. of Phys. Chem. xxviii. p. 673 (1924).
Mukherjee, Proc. Roy. Soc. A, ciii. p. 102 (1923).
(3) Freundlich, *Kapillarchemie*, 3rd ed. p. 335.
Helmholtz, *loc. cit.* (1).
Billiter, *Zs. f. Phys. Chem.* xlviii. p. 542 (1904).
(4) Freundlich, *Kapillarchemie* (loc. cit.).
Dorn, *Ann. d. Phys.* v. p. 20 (1878); ix. p. 513 (1880); x. p. 46 (1880).
Billiter, *Ann. d. Phys.* xi. p. 937 (1903).
Freundlich and Mäkelt, *Zs. f. Elektroch.* xv. p. 161 (1909).

It is at once apparent that there is an intimate relation existing between these phenomena, B being in general the converse of A, and within both A and B, (II.) is the converse of (I.).

The common feature of all these phenomena is illustrated by the explanation which has been offered by the so-called Helmholtz double layer.

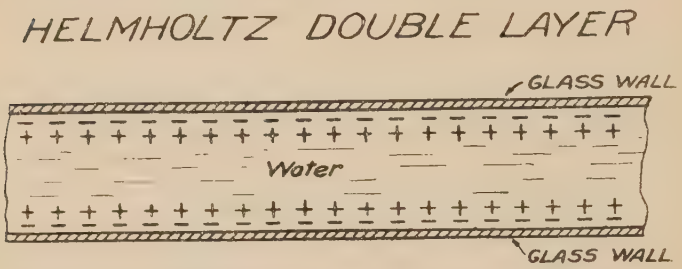
## II. *The Helmholtz Double Layer.*

Historically the discussion of electroendosmose and cataphoresis came first (Wiedemann, *l. c.*). In order to explain these curious motions, Quincke (1) suggested that the solid in contact with the liquid became charged in some way with electricity of one sign, while in the immediate vicinity of the solid wall there was a layer of electrical charges of the opposite kind, exactly equal in magnitude to the first. Helmholtz (1) first offered a mathematical treatment of Quincke's suggestion an l arrived at certain formulæ the truth of which could be—and, in fact, was—tested and proved to be true. The picture presented is consequently known as the Helmholtz

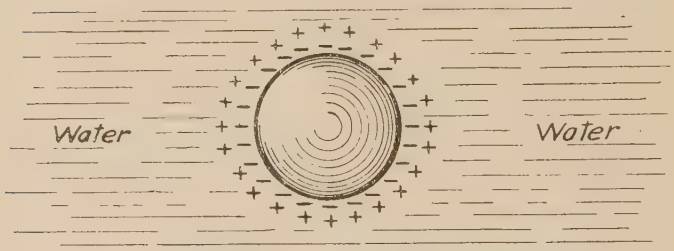
double layer, which was further amplified several years ago by Lamb (1).

When two electrodes, positive and negative, maintain a difference of potential at the ends of the capillary tube, the liquid is dragged through towards the negative electrode and slips as a sleeve through the tube. A similar but rather more fantastic explanation was offered of the movement of solid particles through stationary liquid media.

Fig. 1.




ELECTROENDOSMOSE IN CAPILLARY TUBES




PARTICLES IN WATER

The assumption adopted by both Helmholtz and Lamb, in which they have been followed by others, is that the total thickness of the double layer is just of molecular dimensions, *e.g.*, it is assumed that the charged particle plus its double layer gives an entity which is electrically neutral. This assumption of the thickness of the layer is not implied in the Helmholtz test formula for electroendosmose, which is as follows * :—

$$v = \frac{r^2 \epsilon \mathrm{DE}}{4\eta l},$$

* Freundlich, *Kapillarchemie*, 3rd ed. p. 329.

where

$v$ = volume of liquid transported per sec. through a tube of radius $r$ and length $l$ ;
$\eta$ = coefficient of viscosity of the liquid ;
D = dielectric constant of liquid ;
$\epsilon$ = difference of potential between stationary charged layer and the outer Helmholtz layer ;
E = the electromotive force applied to electrodes at the ends of the capillary tube.

Consequently, the fact that this formula has been found to hold true for various experimental tests does not demand one's assent to the assumed thickness of the outer layer.

Comparatively recently, the theory of the double layer has been dealt with quite fully by Gouy *, Chapman †, and Debye and Hückel ‡.

In discussing the distribution of charges at the surface of contact of a metal electrode and water, Gouy takes for granted that there is a surface charge distributed over the metal electrode, but with regard to the Helmholtz double layer he remarks that "the outer layer of charge cannot be purely superficial, but the ions of this layer are subjected to two sets of forces, those electrical tending to make them accumulate in a layer immediately contiguous to the electrode and those of diffusion, which tend to re-establish the original conditions, *i. e.* uniform distribution of positive and negative ions." For example, if the solid wall is negatively charged, there will be a distribution of negative electricity immediately on the solid wall ; this will tend to attract to the immediate neighbourhood a counterbalancing number of positive ions and repel the negative ions. However, diffusion forces intervene so that, if one could proceed from the wall into the liquid, one would pass through a layer rich in positive ions, poor in negative but not entirely devoid of negative, and, as one got further and further from the wall, the distributions of positive and negative ions would tend to become equal. Defining the distance of the centre of gravity of the outer layer as the distance at which an infinitely thin layer of ions equal in total charge to the outer Helmholtz layer must be placed in order that it should exert the same electrical force as the real distribution, Gouy deduced a formula for this distance, which serves as an index of the distribution of this outer layer. A numerical calculation for

* Gouy, *Jour. de Phys.* (4) ix. p. 457 (1910).
† Chapman, Phil. Mag. xxv. p. 475 (1913).
‡ Debye and Hückel, *Phys. Zeit.*, Feb. 1, 1924.

a very likely distribution of electricity on a solid wall, viz. 10 e.s. units of charge per sq. cm., gives for the distance of the centre of gravity * :

For $\frac{N}{10}$ solution of salt :—

$$0{\cdot}96\ \mu\mu = {\cdot}00096\ \mu = 9{\cdot}6 \times 10^{-8}\ \text{cm.}$$

„ $\frac{N}{100}$ solution of salt :—

$$9{\cdot}6\ \mu\mu = {\cdot}0096\ \mu = 96 \times 10^{-8}\ \text{cm.}$$

„ pure conductivity water :—

$$1010\ \mu\mu = 1{\cdot}01\ \mu = 1{\cdot}01 \times 10^{-4}\ \text{cm.}$$

Modern opinion consequently has departed from the position that the thickness of the double layer is of the order of one or two molecular diameters.

### III. *The Mutual Action of Suspended Particles.*

In order to explain the phenomenon of cataphoresis, Helmholtz (1), after postulating the double layer, says "On the whole, the algebraic sum of the two (charges on the two layers) equals zero, and the centre of gravity of the complete system, solid particle and surrounding positively charged fluid layer taken together, cannot be moved by the electric forces which arise from the potential fall in the liquid through which the current passes. However, the electric force will tend to bring about a displacement, relatively to each other, of the positively charged fluid layer and the negatively charged particle (or *vice versa*), whereby the fluid layer follows the flow of positive electricity while the positive moves in the opposite direction. If the liquid were a perfect insulator the new position would still be a condition of equilibrium. Since, however, through the displacement of the layers, the equilibrium of the galvanic tension between the solid particle and the liquid is disturbed, and, on account of the conductivity of the liquid, always seeks to restore itself, the original state of electrical distribution will tend to be continually reproduced and so new displacements of the particle with respect to the surrounding liquid will continually occur."

* *Note.*—The limiting concentration of aqueous colloidal solutions of gold, silver, and platinum is of the order of 20 mgm. per 100 cc. If in a platinum solution the particles are spheres of radius $10^{-6}$ cm. the least distance between the particles is 100 times their radius and quite the order of distance at which we should expect the mutual repulsion to become active if Gouy's ideas are correct.

One direct result of the above theory is the suggestion reiterated by most colloidal chemists (see Svedberg*. Porter †) that there cannot be any mutual repulsion between colloidal particles, due to the possession of like electrical charges. But, if we accept the ideas suggested by Gouy, we must admit such electrical repulsion whenever the particles approach one another within a distance represented by twice the distance of the centre of gravity of the outer Helmholtz double layer—a distance which might easily become large compared with the diameter of colloidal particles.

In fact, the suggestion of mutual attraction of particles on near approach to one another has been put forward by Smoluchowski ‡ in his paper on the "Mathematical Theory of Coagulation," an assumption to which he was driven by a denial of the existence of mutual repulsion of such particles if they approach one another sufficiently.

Whether one denies mutual repulsion or not it is an undisputed fact that the particles of a stable colloidal solution do remain apart, although they are executing the Brownian movements and, one would judge, are being driven continuously into collision. Smoluchowski made the following assumption: "The simplest assumption would be perhaps that the particles, on near enough approach to one another, are drawn together by capillary action; that union does not take place under normal conditions might be explained by a protective action of the electrical double layer, *which one can picture to one's self as a sort of shell.* On adding electrolytes, on account of the Freundlich absorption, a partial or complete discharge of the double layer takes place, which destroys its protective action, so that collisions and coalescence ensue. . . . . We shall assume that each particle is surrounded by a protective sphere, so that neighbouring particles execute their proper Brownian movements independently and undisturbed, so long as each one keeps without the protective spheres of its neighbours. But two particles are indissolubly united as soon as one of them pierces the boundaries of the protective region of a neighbouring particle."

A more logical view of the distribution of ions in the Helmholtz double layer gives a far simpler and more acceptable explanation of the facts. Gouy's view of this distribution enables us to look upon single colloidal particles as bearing on their surface an effective charge which cannot

* Svedberg, 'Colloidal Chemistry' (Chem. Cat. Co.), p. 102.

† Porter and Hedges, Trans. Far. Soc. xviii. pp. 1–9, see p. 6.

‡ Smoluchowski, *Zeit. f. Phys. Chem.* xcii. p. 129 (1917).

be regarded as being neutralized by the outer Helmholtz layer until a surface is reached at such a distance from the boundary of the particle that the distribution of positive and negative ions becomes equal to one another—*i. e.*, equal to the concentration in the medium itself. As a consequence, although at comparatively great distances from the particle (say, of the order $1{\cdot}0\,\mu = 10^{-4}$ cm.) the electrical force due to the particle is zero, and, consequently, two particles separated by such distances have no mutual repulsion, whenever two particles do approach one another more nearly they will mutually repel with a force which increases rapidly as the distance apart decreases. This explains why "union does not take place under normal conditions." There remains the question, "Why does the addition of electrolytes allow particles to come together"? In connexion with the effect of added electrolytes on the disposition of ions in the outer Helmholtz layer, Gouy proves (1) that, as the concentration of the ions of given valency is increased, the distance of the centre of gravity of the outer layer decreases, (2) that, for given concentration, ions of uni-, di-, tri-, etc., valency will give progressively thinner outer layers, the effect of the valency increasing exponentially with increasing valencies, and (3) that for liquids very slightly ionized the outer layer occupies a considerable thickness, many microns for pure water and still more for other liquids.

In view of these results we may construct a complete picture of the mechanism by which added electrolytes bring about coalescence, and, also, gain an insight into the abnormal effect of ions of different valencies. Let us replace Smoluchowski's assumptions given above by the following: (1) Particles on near enough approach to one another are drawn together by capillary action; (2) that "union does not take place under normal conditions" is accounted for by the mutual repulsion of the particles due to an effective charge, which becomes active when the particles approach one another within the limits of the diffuse Helmholtz outer layer; (3) added electrolytes, by suppressing the diffuseness of this outer layer, reduce the effective distance at which the repulsive force comes into play; and, finally (4), by a discharge of the particle, the electrolytes destroy the source of the force which kept the particles apart.

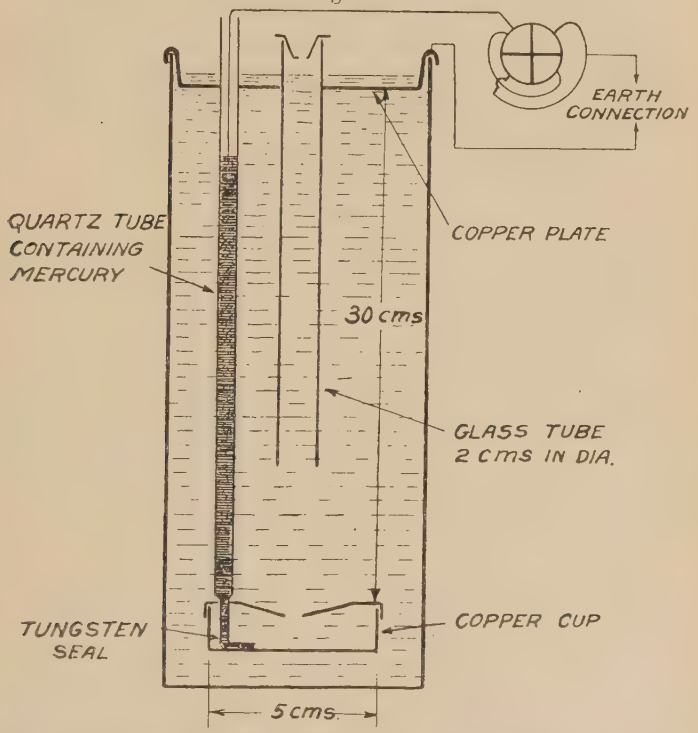
The continuous Brownian movement will tend constantly to cause the particles to collide with one another. While the electrical charge is large enough, mutual repulsion will prevent the particles from approaching one another nearer than a certain distance. If, by the addition of electrolytes, the particles are discharged, which is an experimental fact,

then that mutual repulsion which has prevented collisions is wiped out. Following this out, Gouy's results lead also to a very logical explanation of the curious valency relation existing among ions producing coagulation, known as the Schulze-Linder-Picton law.

### IV. *Experimental Work: Mutual Repulsion of Large Like-charged Spheres falling through Liquids.*

(1) *The Charge on Particles.*—The purpose of the following experiments was to find whether charged particles of macroscopic size moving through liquids could be shown

Fig. 2.




to exert a mutual repulsive effect. As a source of charged particles, advantage was taken of a phenomenon first reported by Dorn (4), viz., that particles of metals, powdered quartz, etc., in falling through a column of water, carry a negative charge and thus tend to set up a difference of potential between the top of the column and the bottom.

These experiments have been repeated and extended using water and other liquids; the collected results are shown in Table II. The apparatus used is shown in fig. 2. A

weighed amount of lead shot (pellets about 1 mm. in diameter) was let fall through a column of liquid and caught at the bottom in a copper cup which was almost entirely closed and joined by means of metallic connexion through a quartz tube with a pair of quadrants of a Dolazelek electrometer. Just below the surface of the liquid in the column an earth-connected copper plate was placed, so that readings on the quadrant electrometer gave an indication of the potential difference set up between the copper plate and the copper cup—or, in other words, a measurement directly proportional to the charge carried down by the lead spheres. Most of the experiments were carried out with water of low conductivity and various other (organic) liquids which invariably had very low conductivity. In carrying out a single observation, the copper cup and its quadrant were isolated by breaking their temporary earth connexion; the needle of the electrometer would then move so as to indicate a very small difference of potential between the copper cup and the upper copper plate; this small deflexion was in any case quite steady. A weighed amount of lead shot was then dropped into the central tube and, of course, fell quickly to the cup. At the instant the lead struck the cup, the electrometer needle swung out, in those cases giving any such effect at all. Central tubes of different lengths were tried, with but little variation shown in the results; consequently, such a length was chosen as to assure that all the shot was caught in the lower cup. Various lengths of liquid columns were tried, and when the distance between the top and bottom electrodes reached some 15 or 20 cm. the results remained constant; a convenient length of about 30 cm. was chosen.

Table II.

| Liquid. | Sign of Charge. | Size of Charge. |
|---|---|---|
| Water ... ..... | — | Large. |
| Alcohol ..... | — | Very small |
| Benzene ..... | — | Very large. |
| Toluene ..... | — | Very large. |
| Xylene ..... | — | Very large. |
| Turpentine ..... | + | Very large. |
| Carbon tetrachloride. | + | Small. |
| Chloroform ..... | No | |
| Ether ..... | No | |
| Acetone ..... | No | |
| Aniline..... | No | |

It will be noticed that benzene, toluene, and xylene give large deflexions showing negative charges carried down, water apparently not so large negative, while turpentine and carbon tetrachloride give positively charged particles (see Wiedemann, 1). It is, of course, apparent that the deflexion of the electrometer will depend on the rate of leak of electricity through the liquid column, so that the deflexion obtained is not an exact measure of the charge carried down by the particles; for a given charge, the deflexion will be greater, the better the insulating properties of the liquid column.

In Table III. are shown the typical results obtained from using increasing weights of shot with the same liquid column. It is apparent that the effect increases with increasing numbers of shot, and we may conclude that each particle of shot carries with it a definite charge.

TABLE III.

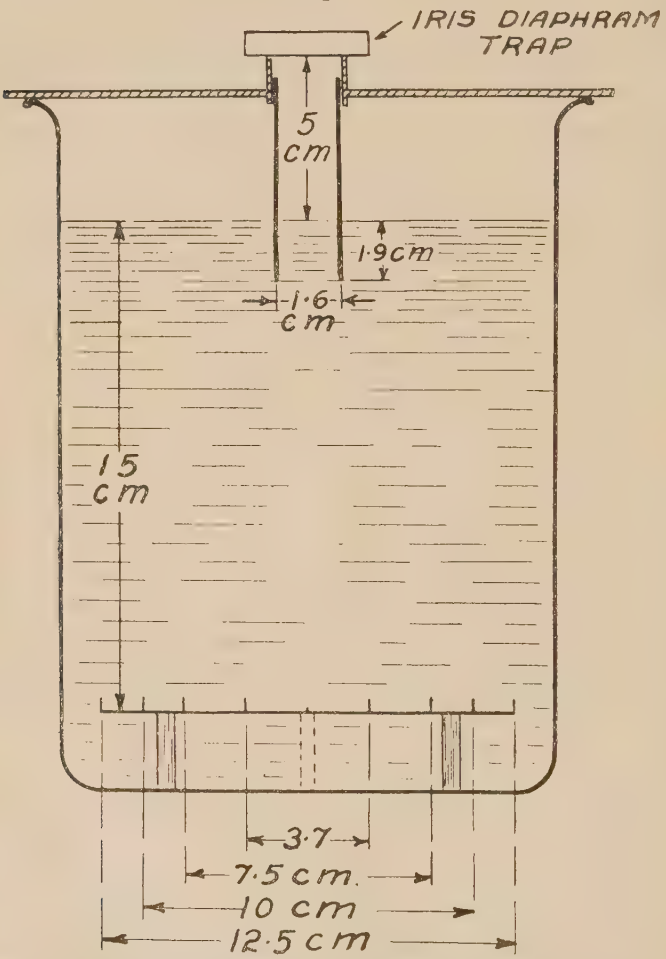
| Weight of Shot. | Electrometer Deflexion. |
|---|---|
| 2 gms. | 2·25 |
| 4 „ | 4·25 |
| 6 „ | 4·85 |
| 10 „ | 7·45 |
| 12 „ | 8·10 |

This phase of the work has not as yet been followed sufficiently to enable one to make a definite announcement of the charge per shot. For the present purpose, we have established that for some media these lead spheres become negatively charged, for others positively charged, while in still other cases there is no indication of a charge at all.

(2) *Effect of Charges on the Scattering of a Cloud of Particles in falling through Liquids.*—If a small cloud of particles such as the lead spheres used above is dropped through a short funnel into a column of liquid, one can easily observe considerable spreading taking place during the descent. Rough preliminary experiments on this phenomenon gave distinct indications that, in cases where the electric charge was large, the spreading was large. Apparatus was devised to attempt to measure the amount of spreading or scattering of the shot on falling through a definite distance in various liquids; measurements have been made whereby a "coefficient of scattering" may be determined.

We are, of course, dealing here with a very complex phenomenon ; the scattering will undoubtedly be affected by hydrodynamic forces due to the spheres moving in parallel paths through a viscous liquid, by possible momentary abrupt

Fig. 3.

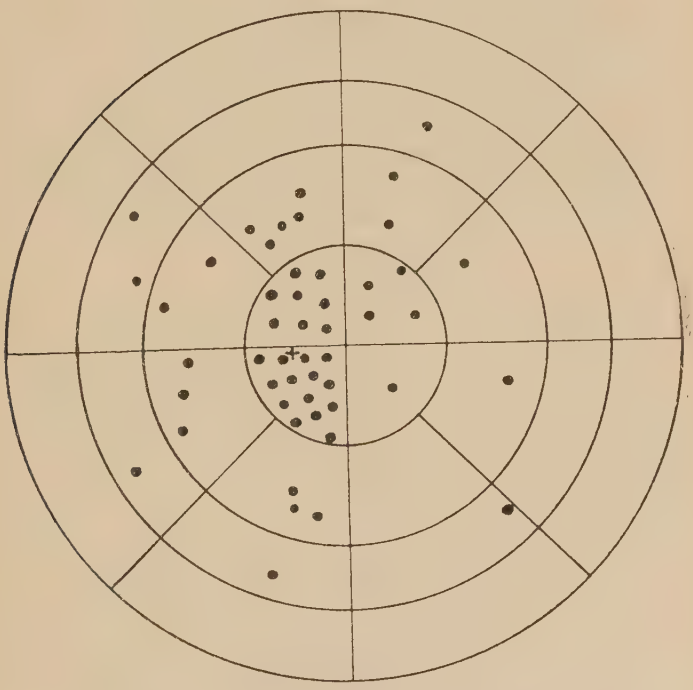



gradients of temperature in the column, by collisions, and so on. However, certain experiments have been made to determine such disturbing features, and definite indications of the scattering due to electrical charges have been found.

The apparatus for determining scattering is shown diagrammatically in fig. 3 ; it represents the final form after

many attempts. For any one observation fifty pieces of shot of fairly uniform size were released simultaneously by a photographic shutter arranged as a trap at the top of a short glass tube kept centrally placed over a vessel of the liquid to be tried. The shot fell a distance of about 5 cm. before striking the liquid, and left the glass tube at a distance of some 2 cm. below the surface. As soon as they leave the

Fig. 4.



*PARTICLES CAUGHT IN TRAY.*

tube scattering occurs. At a fixed distance, 15 cm., below the liquid surface, a flat metal pan, subdivided as indicated in fig. 4, served to catch the pieces of shot and retain them practically at the points at which they struck. This pan, which was lined with filter paper, covered the whole cross-section of the beaker containing the liquid, and so would account for every one of the spheres. As the liquids were transparent the shot positions could be recorded on a plan of the pan (fig. 4).

The radius of the inner circle, which is divided into four quadrants was ·75 in.* ; as, usually, a rather large number of the shot fell in the central part, those in each of the central sections were evenly disposed over each of these sections in the plan made; for all other shot the exact relative position was recorded on the plan. Fig. 4 is a fair sample of what was obtained in hundreds of cases plotted. As is indicated in this figure, it often happened that the centre of greatest density was slightly displaced from the centre of the pan, and consequently in each case a new plan was produced by erasing the bounding lines (fig. 4), selecting a centre of maximum density (marked with a +) and, with this point as centre, drawing a series of concentric circles of 0·5, 1, 1·5, 2·0, 2·5 inches in diameter. The ordinary Gauss formula for departure from a mean was used in determining what would be a measure of the amount of scattering—or a "coefficient of scattering" (S), as it may be called ; this formula is as follows :—

$$S = 0{\cdot}6745\sqrt{\frac{\Sigma x r^2}{N-1}},$$

where $x$ particles are at a distance $r$ from the centre and $N$ = the total number of particles = $\Sigma x$.

For example, in fig. 5, those within the inner circle are taken to be, on the average, 0·25 in. from the centre, those in the next zone average 0·75 in. from the centre, in the next 1·25 in., then 1·75 in., and so on.

From the distribution shown in fig. 5, the coefficient of scattering works out as follows :

$$\Sigma x r^2 = \left\{\begin{array}{l} 16 \times \ \cdot0625 = \ 1{\cdot}00 \\ 18 \times \ \cdot5625 = 10{\cdot}13 \\ 11 \times 1{\cdot}5625 = 17{\cdot}19 \\ \ 5 \times 3{\cdot}0625 = 15{\cdot}31 \end{array}\right\} = 43{\cdot}63.$$

Since $N = 50$, we find $S = {\cdot}629$.

On account of the large quantities of liquids necessary to perform the scattering experiments, only the cheaper, more easily obtained liquids were used for this determination ; at the same time, an attempt was made to cover a wide range of properties in the liquids chosen. In every case, the scattering coefficient given is the mean value of at least six determinations, and may be considered as reasonably

* The inch was used in construction of the pan, and hundreds of calculations were made in this unit ; as the resulting numbers are, in any case, merely arbitrary, the change to cm. has not been made.

reproducible. The results are given in Table IV., in which are recorded the viscosities and certain other calculated scattering coefficients referred to later.

Fig. 5.

*CALIBRATION OF DISTRIBUTION.*

TABLE IV.

Influence of Viscosity of Medium on Scattering.

| Liquid. | Coefficient of viscosity at 10° C. | Viscosity in relation to water as unity. | Scattering coefficient calculated from viscosity curve. | Actual value of scattering coefficient. | Charged? | % Difference. |
|---|---|---|---|---|---|---|
| Water ..... | ·0131 | 1·00 | ·580 | — | Yes | — |
| Turpentine ... | ·0178 | 1·36 | ·510 | ·555 | Yes | +8·8 % |
| Ethyl alcohol. | ·0145 | 1·10 | ·565 | ·582 | Yes | + 3 % |
| Ether ..... | ·0026 | 0·20 | ·750 | ·620 | No | − 17 % |
| Benzene ..... | ·0076 | 0·58 | ·670 | ·686 | Yes | +2·4 % |

As noted above, there must be many influences affecting the amount of scattering ; whatever the influences may be, they would involve in an intimate manner the viscosity of the liquid. In order to test the effect of viscosity, a series of aqueous sugar solutions of varying concentrations were prepared and their scattering coefficients determined. Table V. shows these results, which are illustrated in the curve of fig. 6.

TABLE V.

Scattering Coefficient of Aqueous Sugar Solutions.

| Viscosity. | Specific conductivity. | Scattering coefficient. |
|---|---|---|
| Water taken as 1. | | |
| 1·127 | $6{\cdot}7\times10^{-6}$ | ·559 |
| 1·385 | $2{\cdot}1\times10^{-5}$ | ·507 |
| 1·603 | $2{\cdot}5\times10^{-5}$ | ·444 |
| 1·885 | $2{\cdot}7\times10^{-5}$ | ·395 |
| 1·913 | ..... | ·398 |

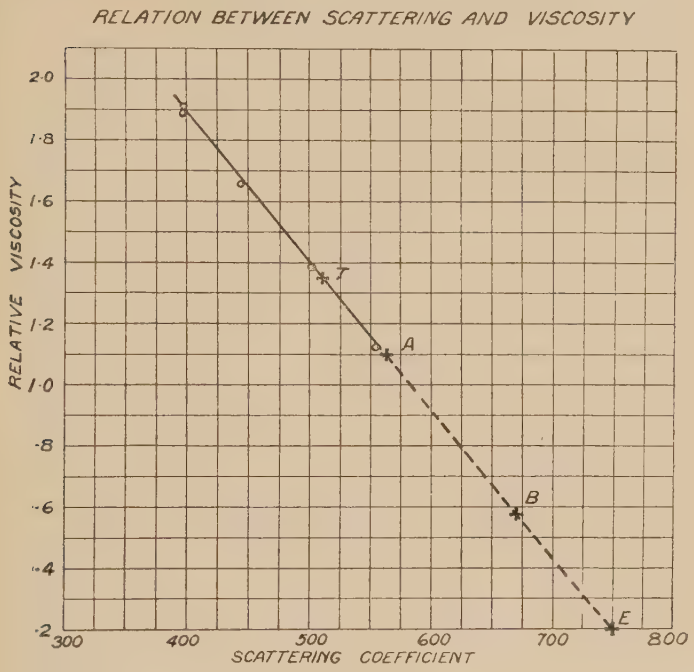
In considering these results it must be borne in mind that viscosity is not the only variable and that a change in conductivity of the solutions may have an effect on the coefficient of scattering. However, the curve shows distinctly that the increase of viscosity cuts down the amount of scattering in a manner almost directly proportional to the viscosity. Judging from this curve, if viscosity were the dominating feature with various liquid media, the coefficients of scattering for the liquids listed in Table V. should be those recorded in column 4 of that Table (points T, A, B, and E in fig. 6). Reference to Table II. will show that those liquids which show a charge produced on the lead spheres have a larger scattering coefficient than sugar solutions. But that liquid which gives absolutely no indication of a charge imparted to the spheres, viz. ether, gives a scattering much less than one would have expected from its very low viscosity. We conclude then that something causing scattering is lacking in the ether but present in the other liquids, and we know that electrical charges are lacking in ether but present in the others.

## V. *Conclusions.*

We believe we have recorded above definite experimental evidence of mutual repulsion produced between similarly charged spheres in various liquids. When one calculates the repulsive force which might be expected, one finds that it

must be very small, but even a very small force between the particles would be sufficient to show a distinct effect on the scattering coefficient.

Fig. 6.




*Summary.*

1. Experiments of Dorn and others on the charge carried by solid particles falling through a column of liquid have been repeated and extended to liquids other than water or aqueous solutions.
2. Lead shot dropped through water, alcohol, benzene, toluene, and xylene carry down a negative charge; through turpentine and carbon tetrachloride a positive charge; and through ether, no charge.
3. Distinct evidence of a scattering due to the possession of a charge has been found, showing that there is undoubtedly a mutual repulsion existing between the particles of shot.

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XXIII. *The J-Phenomena and the Quantum-Theory of Scattering of X-radiation.* *By* R. T. Dunbar, *Ph.D.*, *University of Edinburgh* *.

*Introductory.*

WHEN X-rays pass through matter it is observed that the intensity of the transmitted X-rays, as measured by ionization or other process, is less than the incident intensity, showing that the radiation loses energy. The loss, however, in certain cases has been found to be completely accounted for by the appearance of three secondary radiations, the energy thus emitted being within the error of experiment equal to the energy lost in the process of absorption. The theory of the latter process has been arrived at by detailed examination of the secondary radiations, namely, (1) the "scattered" radiation, (2) the "corpuscular" radiation, or emission of electrons, and (3) the "fluorescent" X-radiation, and it accounts for these, as is well known, by assuming two simultaneous but totally different and independent types of absorption. The incident radiation does work on each electron it passes over, the electric vector producing an acceleration, or more definitely, a forced vibration which conforms with the classical laws of electrodynamics. The energy thus absorbed is re-radiated by the electrons, again according to the classical laws. The other process of absorption is that which we associate with the photo-electric effect. We know nothing of the actual process of absorption but can account for its effects by the assumption of certain empirical laws. The theory is that only those electrons which require less energy to remove them from their atoms than a quantum of the incident radiation can take part in this absorption. The chance that an electron satisfying this condition will absorb energy is almost infinitesimal, and is a function of the energy-level of the electron, of the energy quantum of the incident radiation, and of the atomic number of the element. The energy absorbed by one electron is a single quantum of the incident radiation: part of this energy is required to remove the electron from its atom, and the remainder appears as kinetic energy. Taking this fact in conjunction with the above law, we see that an electron must be ejected for every quantum of energy absorbed. The replacement of electrons thus removed gives rise to X-radiation characteristic of the element. Thus we

* Communicated by Prof. C. G. Barkla, F.R.S.

account for the secondary corpuscular and fluorescent X-radiations.

The theory outlined above is very substantially supported by a comprehensive and detailed examination of the phenomena which accompany the absorption of X-radiation. In particular, the scattered radiation has been found * to have the same penetrating power as the absorbed radiation, and its intensity, distribution, and polarization have been shown to be exactly that which the theory would predict. The evidence is so complete, in fact, that there can be no doubt that the true explanation of the origin of the scattered radiation, in the case of X-rays of long and medium wave-lengths at least, does not differ fundamentally from that which has been given.

Recent experiments by A. H. Compton and others on the scattering of monochromatic radiations of short wave-length by light substances have demonstrated, however, that the process of scattering under certain circumstances must undergo a radical change as the wave-length is reduced below a certain value. Compton † has shown (and his results have been confirmed by numerous other experimenters) that the bulk of the scattered radiation from certain light substances may differ in wave-length from the absorbed radiation, the wave-length being greater in the case of the scattered radiation by $0{\cdot}0242\ (1-\cos\theta)$ Å, where $\theta$ is the angle between the primary radiation and the direction of scattering. The quantum-theory of scattering which Compton ‡ has put forward to account for his results and for the results obtained by various experimenters on the absorption of $\gamma$-rays, in spite of its apparently highly improbable assumptions, has succeeded in accounting remarkably well for the observed phenomena in the region of short wave-lengths. If we accept Compton's theory as a possible process of scattering §, we are led to certain interesting and important considerations. To explain the phenomena of polarization, the reflexion, without change of wave-length, of X-rays by crystals and the presence of a certain intensity of untransformed scattered radiation in Compton's experiments, we are compelled to accept the classical theory as also a possible process of scattering. We have now two possible processes of scattering, and which of the two will be

* Barkla, Bakerian Lecture, 1916.

† A. H. Compton, Phys. Rev. xxi. .p. 483 (1923).

‡ A. H. Compton, *loc. cit.*

§ The application of the term "scattering" to a new process is open to criticism, but in the present paper it should give rise to no ambiguity.

effective must be determined by the conditions of experiment. That there really are two alternative processes seems to be indicated by the fact that experimenters, presumably working under slightly different, though in some cases apparently identical conditions, have obtained results which are clearly inconsistent with one another. Clark and Duane * in attempting to reproduce Compton's results in analysing the radiation scattered by various substances with a highly refined X-ray spectrometer, found that the radiation was unchanged in wave-length. Barkla and Sale †, on comparing the qualities of the primary and secondary rays in the case of paper with heterogeneous rays over a long range of penetrating powers, found at the softer end of the range that the qualities were identical, while at the harder end the qualities were sometimes found to be alike and sometimes to differ by an amount which was considerably greater than that calculated on the theory of Compton. (With heterogeneous rays we have to consider, of course, the possibility of unequal scattering of the constituents which in itself would cause a difference in penetrating power of the beam as a whole.) Assuming, then, the possibility of both processes of scattering and that the condition which decides which process is to be operative depends on some other factor than the wave-length, then for the case of an observer who found the radiation unchanged in wave-length we simply suppose the conditions were not favourable for the quantum process. If we supposed it possible under certain conditions for both processes to be effective simultaneously, an increase in the absorption-coefficient would appear, and this at first sight seems to suggest a solution of the elusive and still unexplained J-absorption phenomena of Barkla. The most important aspect of the present state of the theory of scattering is that we have, or seem to have, a classical process giving place to a quantum process under conditions which have not yet been defined. The discovery of these conditions may well be the first step towards the solution of the more general problem, namely, the process underlying the quantum theory of radiation in general.

When we come to consider what the controlling conditions may be, we are confronted with an enormous number of possibilities. One primary radiation may differ from another in total intensity, in the intensity distribution with reference to wave-length, in polarization, and possibly in other ways which

* Clarke and Duane, P. N. A. S. Mar. 1924.
† Barkla and Sale, Phil. Mag. April 1923.

we are as yet unable to define. Radiators may differ in the degree and the orientation of their crystallization, the strong electrostatic field in the neighbourhood of the X-ray tube may have some influence on the interaction of the X-rays with the material of the radiator, if the latter is an imperfect conductor and is not shielded. There is little to be gained by enumerating possibilities. A solution of the problem should be arrived at most quickly by comparing the conditions obtaining in experiments which gave inconsistent results.

The experiments described below on the absorption of X-rays and on the corpuscular emissions from gases and solids under the action of X-rays were carried out some time ago. Their purpose was to confirm and extend the evidence for the J-discontinuities which had been obtained by previous experimenters in this laboratory. The J-discontinuities did not appear, although others had obtained irregularities of the order of 50 per cent. and more in some cases, irregularities which could not reasonably be accounted for by experimental error or imperfect apparatus. Experiments on the J-discontinuities have therefore produced the same kind of inconsistencies in the same region of the X-ray spectrum as those on the quality of the scattered radiation, and it is natural to suppose that the two phenomena are connected and that the solution of one will provide also the solution of the other. The scattering experiment described at the end of this paper, when taken in conjunction with the results of the absorption and corpuscular experiments, furnishes a good deal of evidence in favour of the quantum theory of scattering. Some attention has been given to the description of the experimental conditions so that these may be available for comparison with those of experiments which gave different results.

## *Experiments.*

The source of X-rays in all the following experiments was a Coolidge tube of the "broad-focus" type with tungsten anticathode. The general radiation from this tube was filtered down with absorbing sheets of aluminium (and sometimes copper) so as to obtain a fairly homogeneous "end-radiation." The thickness of the filter has in all cases been entered in the tables of observations. The source of high-tension current was a "Mammoth" 16 in. coil operated by a "Sanax" mercury break, the primary current being obtained from the City Supply at 230 volts.

I. *Absorption Experiments.*

When a beam of homogeneous rays of intensity $I_0$ passes through a thickness $x$ of a substance of density $\rho$, the transmitted intensity I is given by the well-known absorption formula,

$$I = I_0 e^{-\mu x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

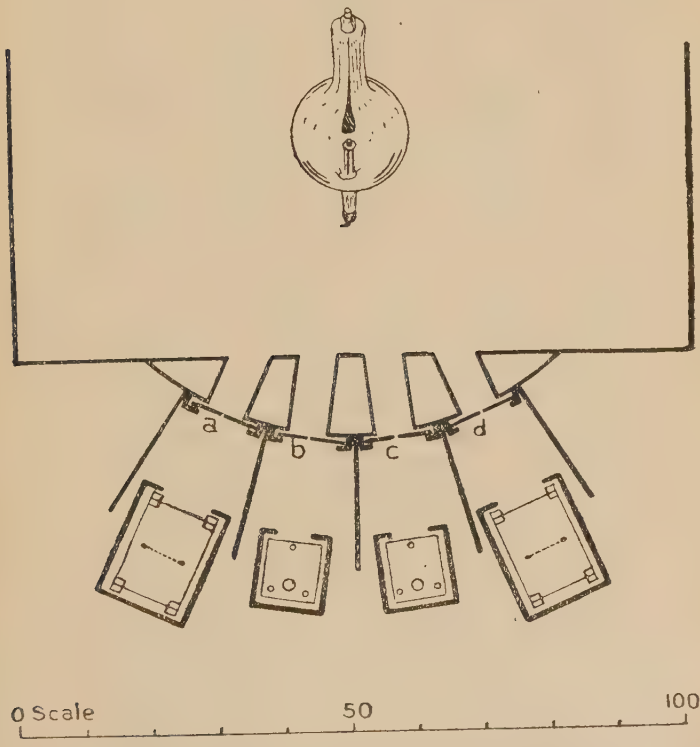
where $\mu$ is a constant which depends on the wave-length of the radiation and on the absorbing substance, and is independent of the thickness $x$. $\mu$ is called the absorption-coefficient and $\mu/\rho$ the mass-absorption coefficient for the absorbing substance for the radiation of that frequency. When the radiation is heterogeneous the value obtained for $\mu$ is found to diminish with increasing thickness $x$, for the less penetrating constituents of the radiation are reduced more rapidly in intensity as the radiation traverses successive layers of the absorbing substance. If $x$ is always adjusted so as to reduce the incident intensity to 50 per cent., the resulting value of $\mu$ will give what we may call an average absorption coefficient. The values of $\mu/\rho$ in the present absorption experiments were found to vary little with $x$, showing that the filtered radiation was fairly homogeneous. This allowed considerable latitude in the percentage absorbed. The percentages actually absorbed have been entered in the tables of observations.

The mass-absorption coefficients for silver, aluminium, and paper were compared with those for copper over a range of wave-lengths which included those at which appeared the J-discontinuities of Barkla and White *. The apparatus was arranged as in fig. 1, with slight modification. Three electroscopes of the Wilson cubical type were set up in front of apertures, so that a pencil of radiation was directed into each. In reality a number of pencils were arranged to enter each so as to ensure a fairly uniform distribution of ionization, and thus reduce the chance of recombination. Filtering sheets of copper and aluminium were placed against the apertures in the lead box $a$, $b$, and $c$, on the side remote from the tube. Their number was increased or diminished so as to give as homogeneous an end-radiation as possible, consistent with a reasonable transmitted intensity. Stray lead L-radiation from the inside of the box was effectively prevented from reaching the electroscopes by the lead cylindrical tubes placed immediately behind the apertures, as shown in the figure. A range of wave-lengths was

* Barkla and White, Phil. Mag. Oct. 1917.

obtained by adjustments of the primary current of the induction-coil, and the intensity was controlled by modifying the temperature of the cathode. The relative rate of leak of the electroscopes when identical filters were placed in front of the apertures was found to vary slightly with wave-length, but this was taken into account. Absorption sheets of copper and other substance were added to the filters at *b* and *c*, respectively. Their thicknesses were adjusted to reduce the

Fig. 1.




rates of leak of the corresponding electroscopes to about half, the third electroscope being used as a standard. The values of the mass-absorption coefficients for the two substances were calculated from equation (1), the rate of leak of an electroscope relative to that of the standard being taken as a measure of the intensity transmitted towards that electroscope. The results of the observations are given in Tables I., II., and III. Plotting the values of $\mu/\rho$ for silver, aluminium, and paper against $\mu/\rho$ for copper (Graph I.), we

TABLE I.—Mass-absorption coefficients for Silver and Copper.

| Filter. | Per cent. absorption by Ag. | Per cent. absorption by Cu. | $\mu/\rho$ for Ag. | $\mu/\rho$ for Cu. |
|---|---|---|---|---|
| 0·2 mm of Cu and 6·5 mm. of Al | 55 | 51 | 13·65 | 4·35 |
| | 57 | 53 | 14·5 | 4·60 |
| | 59 | 55 | 15·1 | 4·81 |
| | 59 | 56 | 15·25 | 4·95 |
| | 60 | 57 | 15·6 | 5·04 |
| | 60 | 57 | 15·8 | 5·10 |
| | 61 | 57 | 16·05 | 5·17 |
| | 62 | 59 | 16·6 | 5·35 |
| | 62 | 58 | 16·8 | 5·32 |
| | 63 | 60 | 16·9 | 5·47 |
| | 63 | 59 | 17·2 | 5·46 |
| | 65 | 61 | 17·8 | 5·78 |
| | 65 | 62 | 18·2 | 5·94 |
| | 66 | 62 | 18·5 | 5·85 |
| | 70 | 67 | 20·45 | 6·75 |
| | 46 | 44 | 11·38 | 3·52 |
| | 48 | 46 | 12·07 | 3·78 |
| | 51 | 49 | 13·04 | 4·10 |

TABLE II.—Mass-absorption coefficients for Paper and Copper.

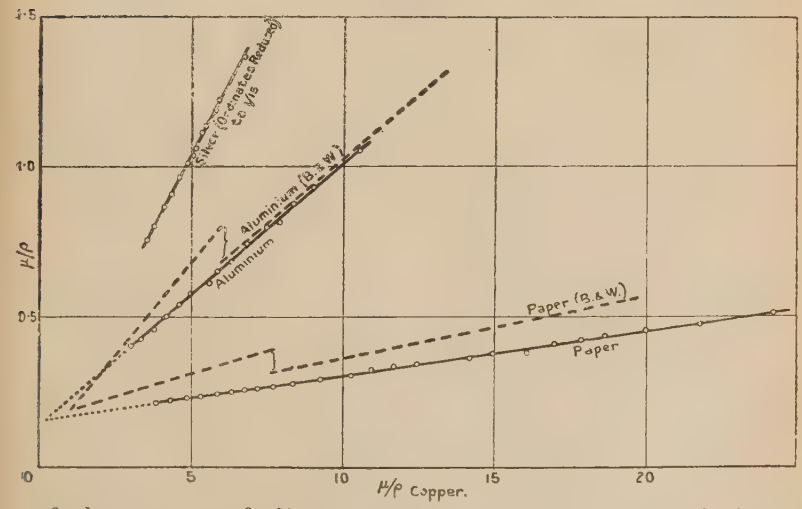
| Filter in mm. of Al. | Per cent. absorption by paper. | Per cent. absorption by Cu. | $\mu/\rho$ for paper. | $\mu/\rho$ for copper. |
|---|---|---|---|---|
| 22·8 | 46 | 48 | 0·218 | 3·82 |
| ... | 47 | 50 | 0·226 | 4·30 |
| ... | 48 | 56 | 0·232 | 4·85 |
| ... | 49 | 58 | 0·239 | 5·30 |
| 18·0 | 50 | 62 | 0·247 | 5·86 |
| 16·3 | 51 | 65 | 0·253 | 6·32 |
| ... | 52 | 68 | 0·260 | 6·77 |
| ... | 52 | 64 | 0·265 | 7·18 |
| 13·0 | 53 | 56 | 0·270 | 7·69 |
| ... | 49 | 42 | 0·282 | 8·37 |
| 10·7 | 40 | 32 | 0·294 | 9·28 |
| ... | 41 | 35 | 0·310 | 10·28 |
| 9·8 | 44 | 37 | 0·328 | 10·95 |
| 7·4 | 44 | 38 | 0·330 | 11·68 |
| 6·5 | 46 | 40 | 0·346 | 12·45 |
| ... | 47 | 45 | 0·365 | 14·20 |
| ... | 49 | 46 | 0·381 | 14·95 |
| ... | 49 | 50 | 0·385 | 16·10 |
| ... | 49 | 49 | 0·412 | 17·00 |
| 4·15 | 50 | 53 | 0·426 | 17·86 |
| ... | 48 | 54 | 0·440 | 18·65 |
| ... | 45 | 57 | 0·457 | 20·00 |
| ... | 46 | 60 | 0·476 | 21·80 |
| 3·25 | 60 | 64 | 0·511 | 24·27 |
| ... | 61 | 66 | 0·544 | 26·05 |
| ... | 62 | 68 | 0·562 | 27·20 |
| nil | 54 | 69 | 0·583 | 28·20 |
| 3·25 | 60 | 72 | 0·604 | 30·45 |

TABLE III.

Mass-absorption coefficients for Aluminium and Copper.

| Filter. | Per cent. absorption by Al. | Per cent. absorption by Cu. | $\mu/\rho$ for aluminium. | $\mu/\rho$ for copper. |
|---|---|---|---|---|
| 0·28 mm. Cu & 16·3 mm. Al | 51 | 47 | 0·402 | 3·00 |
| ..... | 52 | 50 | 0·426 | 3·33 |
| ..... | 56 | 55 | 0·460 | 3·78 |
| 0·28 mm. Cu & 13·0 mm. Al | 59 | 58 | 0·504 | 4·175 |
| ..... | 62 | 69 | 0·542 | 4·60 |
| 0·28 mm. Cu & 9·8 mm. Al | 64 | 65 | 0·578 | 4·98 |
| ..... | 66 | 69 | 0·618 | 5·59 |
| ..... | 52 | 52 | 0·655 | 5·86 |
| ..... | 70 | 74 | 0·686 | 6·32 |
| ..... | 57 | 58 | 0·747 | 6·83 |
| 0·28 mm. Cu & 6·5 mm. Al | 60 | 61 | 0·802 | 7·49 |
| 0·28 mm. Cu & 3·25 mm. Al | 52 | 48 | 0·818 | 7·90 |
| ..... | 58 | 58 | 0·878 | 8·37 |
| ..... | 56 | 53 | 0·933 | 9·00 |
| ..... | 60 | 59 | 1·052 | 10·52 |

Graph I.




find no trace of discontinuities. An examination of the results will show that, by allowing a probable experimental error of only 1 per cent. in the determination of $\mu/\rho$ all

the deviations from a linear relationship between the mass-absorption coefficients can be accounted for. The results obtained by Barkla and White for paper and aluminium are indicated by broken lines in the same graph. A comparison of the two sets of curves shows conclusively that whatever produced the discontinuities in the one set of curves was certainly not present in the other. Hence we may conclude that under the conditions of experiment described above the J-discontinuities do not appear.

The present experiments differ from those of Barkla and White in the following respects :—

(1) A Coolidge tube was used in place of a gas-filled tube.
(2) The distance between anticathode and filters was about 48 cm., or about 2 or 3 times that in the other case.
(3) The anticathode was of tungsten in place of platinum.
(4) The filters were thicker in the experiments now described, about 2 to 4 times as thick in the regions of the discontinuities, so that the end-radiation was very probably more homogeneous.

## II. *Experiments on Corpuscular Radiation.*

The intensity of the corpuscular radiation from a substance exposed to the action of X-rays may be conveniently measured, if the substance is in the form of a gas, by finding the intensity of ionization produced by the complete absorption of the corpuscles by the gas itself. If the substance is in the form of a solid, a plate of the substance may be exposed, in an atmosphere of hydrogen, to the action of the X-rays, when the ionization of the hydrogen will be produced almost entirely (over 99 per cent. probably) by the corpuscles sent out from the plate. The intensity of ionization of the hydrogen will afford a measure of the intensity of the corpuscular emission from the substance of the plate. This method of measurement is evidently suitable for the location of discontinuities. These two methods of studying the intensities of the corpuscular emissions were applied to various substances.

The same ionization-chamber was used throughout the experiments. It consisted of a brass cylindrical tube, 14 cm. long and 10 cm. in diameter. The inside of this tube was covered with aluminium foil, and aluminium plates 1·5 mm. thick were fitted to the ends. The electrode was connected to a Wilson electroscope of the cubical type placed immediately under the ionization-chamber. The necessity for taking into account the loss of corpuscles at the sides, and the loss and gain at the ends of an ionization-chamber,

and the method of correcting for these by taking ionization-pressure curves are sufficiently well known and need not be considered here. For the present purpose the correction for boundary effects was considered unnecessary, the J-effects being much more discontinuous and of a much greater magnitude than the effects produced by loss and gain of corpuscles at the walls of the chamber.

The apparatus was set up as in the previous experiments (see fig. 1). The ionization-chamber on the left was normally not in use. It was used later to make a direct comparison between pairs of substances where an indirect comparison in terms of a standard electroscope was not considered sufficiently conclusive. One electroscope was used as a standard, the other to measure $\mu/\rho$ for the incident radiation, aluminium being used as absorbing substance. The two gases whose ionizations were compared with that of air were sulphur dioxide and ethyl bromide. The ends of the two electroscopes and of the ionization-chamber were covered on the inside with five layers of filter-paper, and precautions were taken to ensure that the radiation was in no case incident on the electrodes. The three gases, air, sulphur dioxide, and ethyl bromide, were introduced in turn into the ionization-chamber, and at least one complete set of observations over a considerable range of wave-lengths was taken with each gas. The ratio of ionization in the chamber, when filled with air, to that in the standard electroscope was found to be constant to about one per cent. over the whole range, from which it was concluded that the difference in the qualities of the pencils of radiation entering the standard electroscope and the ionization-chamber was negligible. In the case of $SO_2$ several sets of observations were taken, the conditions being altered slightly from set to set. Well-filtered radiation was tried, and a set was taken with the filtering much reduced. The readings were necessarily rough, for reasons that need not be entered into, the accuracy being of the order of about three or four per cent.; but this degree of accuracy would have been ample to show the presence of the J-effect had it been present to only 15 per cent. of its value in the experiments in which it had previously appeared. Table IV. gives the results obtained when the radiation was not highly filtered. In Barkla's experiments, with diminishing wave-length the ratio $\frac{\text{Ionization of } SO_2}{\text{Ionization of Air}}$ decreased suddenly and later increased suddenly to its original value, and the explanation given was that the corpuscular emision from air increased in association with a J-absorption, and, at a shorter

TABLE IV.

Relative Ionizations of Sulphur Dioxide and Air.

| Filter in mm. of Al. | $\mu/\rho$ (Al). | Per cent. absorbed by Al. | $\frac{SO_2}{Air}$ (observed). | $\frac{SO_2}{Air}$ (corrected) *. |
|---|---|---|---|---|
| 9·8 | 0·60 | 41 | 0·923 | 0·932 |
| ... | 0·727 | 47 | 0·978 | 0·991 |
| ... | 0·92 | 55 | 0·994 | 1·012 |
| 3·25 | 1·53 | 54 | 1·007 | 1·041 |
| nil | 4·45 | 69 | 0·847 | 0·966 |
| nil | 4·0 | 65 | 0·853 | 0·951 |
| 9·8 | 0·792 | 50 | 0·986 | 1·001 |
| 1·5 | 2·22 | 44 | 0·929 | 0 976 |
| ... | 2 42 | 47 | 0·912 | 0·962 |
| ... | 2·60 | 49 | 0·908 | 0·967 |
| nil | 4·1 | 41 | 0·853 | 0·955 |
| 3·25 | 1·48 | 53 | 1·009 | 1·041 |
| ... | 1 63 | 56 | 0·995 | 1·031 |
| ... | 1·81 | 60 | 0·994 | 1·034 |
| ... | 1·64 | 56 | 1·005 | 1·041 |
| 4·75 | 1·13 | 44 | 1·011 | 1·036 |
| ... | 1·24 | 47 | 1·014 | 1·043 |
| ... | 1·36 | 50 | 1·014 | 1·044 |

* The ratios in the 4th column are given on an arbitrary scale. The 5th column gives the same ratios corrected for the fact that the X-rays fall off in intensity as they pass through the gas.

wave-length, that from $SO_2$ increased also by the same fraction. No such effect was observed in the present experiments. The results are plotted with the frequency of the incident radiation as abscissæ, which makes the results simpler of interpretation than when the ratios are plotted against $\mu/\rho$, as will appear. To convert values of $\mu/\rho$ for Al to frequencies, it was not considered advisable to use experimental values of the absorption-coefficients of wave-lengths, for these contain experimental errors of appreciable magnitude, judging by the difference between the results of different observers. The probability is that, apart from the regions which contain spectral lines, there exists a smooth relationship between wave-length and absorption-coefficient. An approximate empirical formula (that of Kossel, Siegbahn, and Glocker), founded on experimental values determined up to a recent date, was used, namely,

$$\mu/\rho = 0{\cdot}2 + 0{\cdot}0195\, N^{2{\cdot}58}\, \lambda^{2{\cdot}8},$$

where N is the atomic number of the absorbing element, and $\lambda$ is the wave-length. This formula is only roughly approximate in the case of Al, but points where irregularities appear can have their true frequency or wave-length determined

afterwards, if necessary, by reference to experimentally determined values of the absorption-coefficients of definite wave-lengths.

Graph II. shows the results of the comparison of the ionization of $SO_2$ and of ethyl bromide with that of air. In

Graph II.

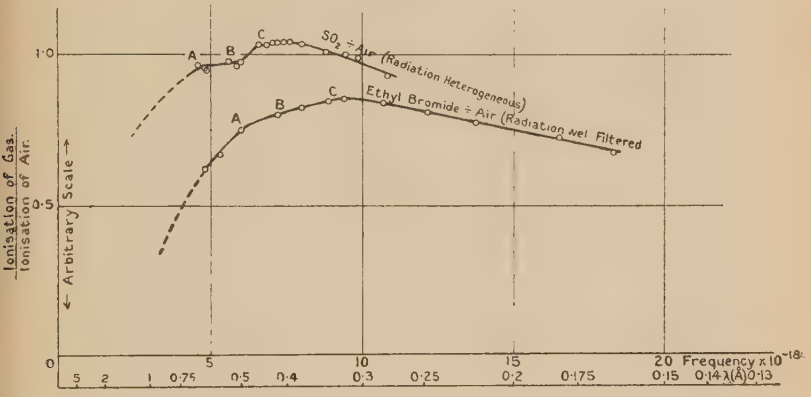



TABLE V.

Relative Ionizations of Ethyl Bromide * and Air.

| Filter in mm. of Al. | $\mu/\rho$ (Al). | Per cent. absorbed by Al. | Eth. Br. / Air (observed). | Eth. Br. / Air (corrected) †. |
|---|---|---|---|---|
| 29·3 | 0·295 | 54 | 0·673 | 0·676 |
| 22·8 | 0·328 | 51 | 0·716 | 0·721 |
| 16·3 | 0·410 | 51 | 0·766 | 0·774 |
| 13·0 | 0·495 | 50 | 0·798 | 0·807 |
| 9·8 | 0·622 | 50 | 0·830 | 0·840 |
| 8·0 | 0·795 | 51 | 0·839 | 0·853 |
| 6·5 | 0·90 | 55 | 0·826 | 0·844 |
| 4·65 | 1·14 | 52 | 0·804 | 0·824 |
| 3·25 | 1·45 | 53 | 0·775 | 0·800 |
| 1·40 | 2·27 | 43 | 0·721 | 0·752 |
| 0·47 | 3·20 | 55 | 0·627 | 0·667 |
| 0·00 | 4·02 | 43 | 0·569 | 0·620 |

* The ethyl bromide vapour was mixed with air, but fully 90 per cent. of the ionization was produced by the corpuscular emission from the ethyl bromide.

† This column is obtained as explained in footnote to previous table.

the case of ethyl bromide the radiation was well filtered (see Table V.). This substance, owing to its low vapour-pressure, could only be tested at a pressure of a cm. or two of mercury. To prevent excessive loss of corpuscles at the

walls of the chamber, which would result from the low pressure of the gas, the pressure was brought up to atmospheric by introducing air. The ionization of a mixture of gases is a complex matter, but for our present purpose it is unnecessary to enter into considerations of this kind. Over 90 per cent. of the ionization of the mixture was attributed to corpuscles from ethyl bromide. Corrections were made on the observations for the fact that the intensity of the primary radiation falls off more rapidly when it passes through the heavier gases. A knowledge of the absorption coefficients of the gases enabled the corrections to be applied.

Before interpreting the features of these curves, let us consider what shape of curve theory would lead us to expect. Absorption experiments have shown that in regions not too near characteristic frequencies of either element there exists a linear relationship between the absorption coefficients of any two elements, and that the "true" absorptions by fixed masses of the two elements (that is, total absorptions less the absorptions associated with scattering) are proportional. The ratio of the "true" absorptions by given masses of two gases is therefore independent of the wave-length in regions remote from their spectral lines. But each quantum absorbed gives by theory one corpuscle; therefore the relative numbers of corpuscles emitted is constant. The energy per corpuscle is $h\nu - h\nu_K$, assuming that the corpuscles emerge from the K-level, which most of them do in the present case. Consequently the total energy of the corpuscular emission and therefore the intensity of the ionization is proportional to $\nu - \nu_K$, so that the ratio $\frac{\text{Ionization of Gas}}{\text{Ionization of Air}}$ is proportional to $\frac{\nu - \nu_{K_g}}{\nu - \nu_{K_a}}$, and therefore to $1 - \frac{\nu_{K_g} - \nu_{K_a}}{\nu - \nu_{K_a}}$, where $\nu_{K_g}$ and $\nu_{K_a}$ are the K-level frequencies for gas and air respectively. This gives a rectangular hyperbola cutting the frequency axis at $\nu_{K_g}$. The parts of the curves to the left of C agree very well with this theoretical result, within the error of experiment. The small irregularities at B may be real, but are covered by experimental error and need not therefore be discussed. The drop in the values of the ratios to the right of C could be accounted for by (1) an abnormal falling off in the ionizations of the gases $SO_2$ and ethyl bromide, or (2) an abnormal increase in the ionization of air. This feature of the curves did not

receive due attention at the time the experiments were carried out, for, from the location and gradual nature of the irregularity, it was clear that it was not connected with the investigation in hand, namely the J-discontinuities. The application of the method of ionization-pressure curves to eliminate the effects of end-losses would have simplified the interpretation of this irregularity. Part of the effect is certainly due to end-losses, for the elements contained in paper are on the whole much lighter than those contained in the two heavy gases, and, as the radiation becomes more penetrating, the corpuscles have greater velocities and the net loss at the ends becomes more important. From our knowledge of the ranges of corpuscles in gases, however, it seems unlikely that so large a fraction of the corpuscles would reach the walls of the chamber as would be required to reduce the ionization to an extent that would explain the irregularity. It would seem reasonable, therefore, to conclude that these results are not inconsistent with the view (to which we are led by the results of experiments on plates) that when the radiation is reduced in wave-length below about 0·4 Å, the corpuscular emission from *Air* begins to show an abnormal increase in intensity. The fact that the beginning of the irregularity (C) appears at different wave-lengths on the two curves is probably accounted for by the heterogeneity of the radiation in the case of the upper curve, for the more penetrating constituents are present when the absorbability is still comparatively great.

The plates experimented on consisted of (1) paper, (2) aluminium, (3) copper, and (4) lead. The ionization-chamber was the same as that used in the experiments on gases. The aluminium plates, 1·5 mm. thick, were retained at the ends. Several thicknesses of filter-paper were fixed to the inside surfaces of these aluminium plates when paper was the substance under examination. (It should be mentioned that five sheets on each end gave the same results at all wave-lengths as twelve sheets, showing that five sheets of paper were sufficient to prevent corpuscles from the aluminium reaching the gas.) The paper was replaced by copper foil, and later, when lead was being tested, the aluminium plates were covered with paper and lead respectively, the lead being at the back. The ionization of the hydrogen gas in the latter case was due almost entirely to the corpuscles from lead. It seemed desirable to have the substance whose corpuscular radiation was being tested at both ends of the chamber, in order to eliminate errors which would arise from the known unsymmetrical

distribution of the corpuscular emission. This was impracticable in the case of lead, which nevertheless gave a curve with the same features as the others, so that the precaution would seem to have been unnecessary. The hydrogen was obtained from a Kipp's apparatus charged with zinc and dilute sulphuric acid. The action of the zinc on the acid was very slow, which would indicate that these substances were fairly pure. The hydrogen was passed through caustic potash, calcium chloride, and cotton-wool, but no other precautions were taken to exclude impurities.

The ionization of the hydrogen by each of the four substances was compared with that in the standard electroscope over a range of wave-lengths. The results are given in Table VI., and the ratios (on an arbitrary scale) are

TABLE VI.

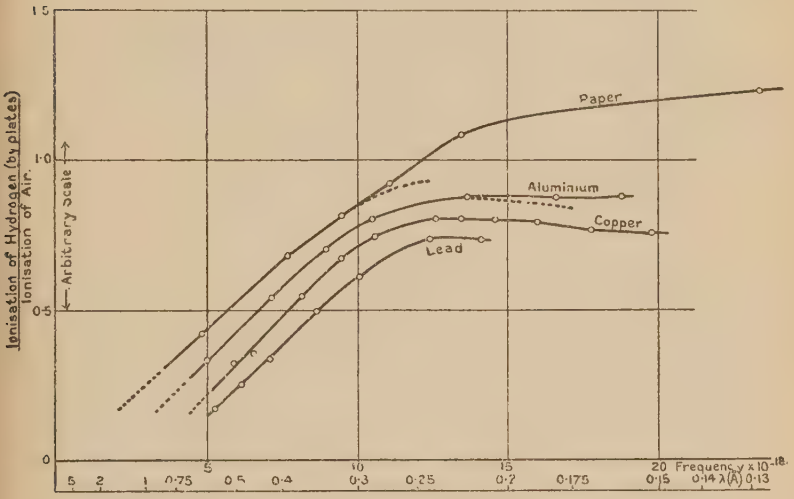
Ionization of Hydrogen by Paper, Aluminium, Copper, and Lead, relative to Ionization of Air.

| Filter in mm. of Al. | $\mu/\rho$ (Al). | Paper * / Air * | Aluminium * / Air * | Copper * / Air * | Lead * / Air * |
|---|---|---|---|---|---|
| 29·3 | 0·244 | 1·232 | | | |
| ,, | 0·275 | ... | | 0·758 | |
| ,, | 0·284 | ... | 0·880 | | |
| ,, | 0·300 | ... | ... | 0·765 | |
| 22·8 | 0·328 | ... | 0·872 | | |
| ,, | 0·336 | ... | ... | 0·792 | |
| 19·6 | 0·376 | ... | ... | 0·801 | |
| 22·8 | 0·392 | ... | ... | ... | 0·736 |
| 16·3 | 0·407 | ... | 0·877 | | |
| ,, | 0·416 | 1·086 | ... | | |
| ,, | 0·422 | ... | ... | 0·804 | |
| ,, | 0·462 | ... | ... | 0·804 | |
| ,, | 0·480 | ... | ... | ... | 0·736 |
| 9·8 | 0·580 | 0·925 | | | |
| ,, | 0·616 | ... | ... | 0·746 | |
| ,, | 0·635 | ... | 0·803 | | |
| ,, | 0·69 | ... | ... | ... | 0·610 |
| 6·5 | 0·79 | 0·812 | ... | 0·672 | |
| ,, | 0·88 | ... | 0·708 | | |
| ,, | 0·955 | ... | ... | ... | 0·497 |
| 3·25 | 1·09 | ... | ... | 0·548 | |
| ,, | 1·26 | 0·683 | | | |
| ,, | 1·48 | ... | 0·543 | | |
| ,, | 1·50 | ... | ... | ... | 0·339 |
| 1·40 | 1·83 | ... | ... | 0·358 | |
| ,, | 2·125 | ... | ... | ... | 0·252 |
| nil | 2·42 | ... | ... | 0·321 | |
| ,, | 3·25 | ... | ... | ... | 0·172 |
| ,, | 3·60 | ... | 0·334 | | |
| ,, | 3·97 | 0·423 | | | |

* The ratios are given on an arbitrary scale.

plotted against the frequency of the primary radiation as abscissæ in Graph III. Commencing from the left, the ratios in every case are seen to increase rapidly with diminishing wave-length. This is to be expected for, whereas in the standard electroscope the corpuscles come from a fixed mass of air, in the case of hydrogen the corpuscles come from a thickness of the substance at the ends of the chamber which increases with the initial velocities of emission. The ratio of ionizations is therefore a function of the initial speeds of the corpuscles, that is of $\nu-\nu_K$ in the case of paper, aluminium, and copper, and $\nu-\nu_L$ in the case of lead, approximately. Hence we find that when the

Graph III.




initial straight parts of the curves are produced backwards, the intersection with the frequency axis is approximately at $\nu=\nu_K$ or $\nu_L$, as the case may be. Proceeding to the right, the rates of increase of the ratio with frequency fall off, and the curves become horizontal or nearly so. The energy of the corpuscles is at this stage incompletely absorbed by the hydrogen, the fastest of them from the surfaces of the plates getting right across the chamber without giving up an appreciable fraction of their energy. Ionization-pressure curves showed that even when the radiation was as soft as 3·0 $\mu/\rho$ Al, which is near the long wave-length limit of our range, a certain number of the corpuscles were getting across the chamber. Clearly,

when the velocities are very great, if we could replace the hydrogen with the substance of the plates in the form of a gas, the pressure being such as to offer the same obstruction to the corpuscles as the hydrogen, the ionization would be the same, for the corpuscles emitted by the gas would reach the walls without giving up an appreciable amount of energy ; and this hypothetical arrangement is a chamber of the same kind as the electroscope, in the sense that it contains a gas with ends properly compensated for loss and gain of corpuscles. We would therefore expect that the ratio :

$$\frac{\text{Ionization due to plate}}{\text{Ionization of air}}$$

would approach, asymptotically from below, the ratio :

$$\text{constant} \times \frac{\text{Energy of corpuscular per gram of substance of plate}}{\text{Energy of corpuscular per gram of air}}.$$

The latter, as has already been seen, when the true absorptions of the substances are proportional, gives a rectangular hyperbola of the form :

$$\text{R (Ratio)} = \text{const.} \times \left(1 - \frac{\nu_{K_g} - \nu_{K_a}}{\nu - \nu_{K_a}}\right).$$

The only curve which approximately approaches such an hyperbola asymptotically on the right is that for paper, and this substance has an "average" atomic weight very nearly the same as air. The other substances show a distinct deficiency of ionization at short wave-lengths, the deficiency beginning to be noticeable at about 0·32 Å (see graph). We are led to conclude that the corpuscular emission from air is increasing abnormally with diminishing wave-length, and that the effect begins to be appreciable at 0·32 Å. A comparison of the curves for aluminium and copper shows that the former begins to show an abnormal increase relative to the latter at about 0·215 Å. To check these two results a direct comparison was made between the substances paper and aluminium, and aluminium and copper, using two identical ionization-chambers, placed symmetrically with respect to the tube (see fig. 1). Both irregularities appeared very distinctly. In the case of the comparison of aluminium and copper it appeared, as anticipated, at $\mu/\rho_{Al} = 0{\cdot}4$, and attained a value of the order of 10 per cent. when the radiation was hardened to $\mu/\rho_{Al} = 0{\cdot}35$. The wave-lengths on the graph were those calculated from an approximate formula connecting absorption-coefficients and wave-lengths. Using the experimental

results of Siegbahn and of Hull and Rice instead of this formula, we get the values 0·405 * and 0·275 Å.† for paper and aluminium respectively.

We must now consider this result from the point of view of energy. The mass-absorption coefficients of paper and copper over this region have been shown to be free from irregularities, within a small error of experiment, under apparently identical conditions of experiment (see Graph I.). Now the "true" absorption by paper is transformed almost entirely to corpuscular emission; so that if the total absorption and therefore the true absorption is normal, and the corpuscular radiation increases abnormally when we decrease the wave-length of the incident radiation, the additional energy converted to corpuscular radiation must necessarily come from the absorption associated with scattering—directly, or by a subsequent transformation of the scattered radiation. (It is assumed that no third independent type of absorption exists.) It is here that Compton's theory of scattering comes to our assistance. His theory requires that part of the "scattered" absorption shall give rise to corpuscular radiation. The absorption associated with the latter is given by

$$\sigma_a = \frac{\alpha}{(1+2\alpha)^2} \times \sigma_0,$$

where $\alpha = 0{\cdot}0242/\lambda$, and $\lambda$ = wave-length of the absorbed radiation in Ångström units, and $\sigma_0$ is the value of the scattering coefficient on the classical theory, which is $0{\cdot}2 \times$ the density of the absorbing substance. The true absorption-coefficients for paper and aluminium for hard X-rays are associated almost entirely with corpuscular emission. The total absorptions associated with corpuscular emission are therefore, according to Compton, approximately proportional to $\bar{\mu}$ (paper) $+\sigma_a$ and $\bar{\mu}$ (Al) $+\sigma_a$ for paper and Al, respectively, $\bar{\mu}$ being the total absorption-coefficient less the total scattering-coefficient. The term $\sigma_a$ is found, by substituting the known values of $\bar{\mu}$, to add about 8 per cent. and 6 per cent. to the latter at the respective wave-lengths 0·405 and 0·275 Å. for paper and aluminium, and in each case the percentage added has just begun to increase very rapidly. The full effect of the additional corpuscular

* The radiation was clearly too heterogeneous and the observations too few in number to show separate discontinuities for nitrogen and oxygen.

† It will be noticed that the values 0·32 and 0·215 obtained by applying the formula are considerably in error.

emission would not at first be manifest, for initially the "scattered" corpuscles have only a small speed, and only a thin surface-layer would contribute to the ionization. With shortening wave-length, however, the speeds increase rapidly. This would explain why the actual irregularities at the wave-lengths given are much less than 8 per cent. and 6 per cent.

Further confirmation of this explanation of the irregularities appears on examining the absorption curves in Graph I. more closely. The straight lines drawn through the observations for paper and aluminium intersect at about $0{\cdot}2\ \mu/\rho$ for Cu. According to the classical theory, all three substances should have a value approximating to 0·2 for the mass-absorption coefficient at this point of intersection, the true absorption and the scattered absorption approaching the limiting values 0 and 0·2 times density, respectively, when the wave-length becomes vanishingly small. The values of $\mu/\rho$ for paper and aluminium at the point of intersection are both about 0·16. Compton's theory corrects the classical coefficient of scattering by applying the factor $1/(1+2\alpha)$, where $\alpha$ is, as before, $0{\cdot}0242/\lambda$ (A.). Assuming for the moment that this is right, if we add to the observed values of the mass-absorption coefficient the quantity

$$0{\cdot}2\left(1-\frac{1}{1+2\alpha}\right),$$

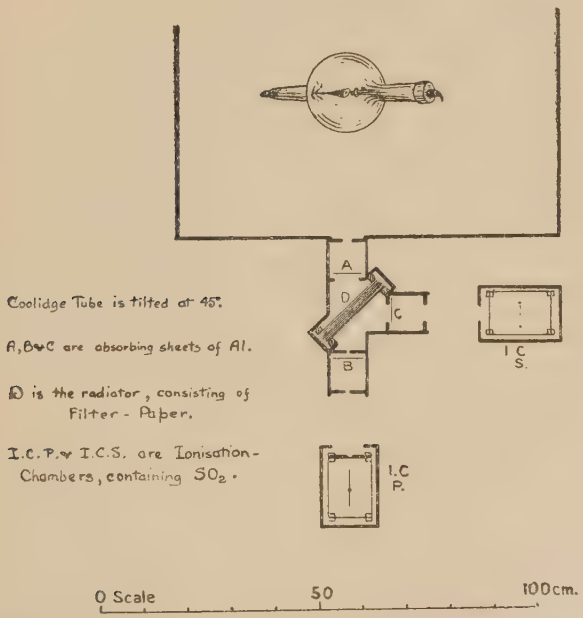
we should obtain the values of the mass-absorption coefficient which would have been obtained had the classical theory been correct. When this is actually done, the new values fall as well upon straight lines as before, but on producing them towards the origin, the three coefficients at the point of intersection are found to be approximately 0·195, 0·195, and 0·2 for paper, aluminium, and copper respectively, and these values are as nearly equal to the classical values as could be expected.

### III. *Scattering Experiment.*

We have been led to the conclusion, in discussing the previous experiment, that certain irregularities which appeared might reasonably be explained on Compton's theory of scattering. From this point of view the excess ionization of hydrogen by paper and aluminium when the wave-length is short is to be attributed to the process of scattering, the electrons which scatter the radiation retaining the vector difference between the momenta of the incident and scattered quanta. If this theory is correct, the energy of the excess corpuscular emission must be taken from the "scattered" absorption, only a fraction of the latter reappearing

as X-radiation. If it were shown by experiment that the energy scattered as X-radiation was less than the energy absorbed in association with scattering, we should then have qualitative proof that we had correctly traced the origin of the excess corpuscular radiation. The agreement already pointed out between Compton's theory and experiment with regard to the total energy absorbed in the process of scattering, in the case of paper and aluminium, leads us to suppose that Compton's calculated value for the absorbed energy is at least very close to the truth. If, then, we measure the quality and intensity of the scattered radiation in a number of directions over a range of wave-lengths, and

Fig. 2.




compare the results with the values calculated on Compton's theory, agreement between experiment and theory would prove that only part of the energy absorbed in the process of scattering reappears as X-radiation. In the present experiment, observations of the quality and intensity of the radiation scattered by paper at right angles to the primary were taken. The intensity was measured in arbitrary units only, for the intensity at the long wave-length end of the range is known to be correctly given by the classical theory.

The apparatus is shown in fig. 2. The cathode stream

is directed at right angles to the line between anticathode and radiator, but the angle between the cathode stream and the pencil of radiation scattered at right angles into the ionization-chamber was 45° (see figure). This ensured that polarized and unpolarized primary rays would be scattered in the same proportion in the direction of the ionization-chamber. This precaution is obviously essential when the radiation is appreciably heterogeneous, for the constituents of the general radiation from an X-ray tube are not polarized to the same extent. The sheets of filter-paper used as radiator were set up so that their normal bisected the angle between the direction of incidence of the primary and the direction of the secondary scattered pencil, ensuring equal absorption by the radiator of primary and secondary, assuming they differ little in quality.

To measure the intensity of primary and secondary, two ionization-chambers of the same dimensions were used—those, in fact, which were used in the experiments described above. To simplify the interpretation of the results, the same gas was used in both at the same pressure—sulphur dioxide at atmospheric pressure. The windows were of aluminium, 0·1 mm. thick. The primary pencil of radiation entering the chamber was about 0·5 mm., and the secondary about 3 cm. in diameter. Precautions were taken to ensure "saturation," particularly in the case of the primary, where the ionization was produced by a fine intense pencil of radiation in a comparatively large chamber filled with a heavy gas. Care was taken also to prevent incidence of X-radiation on the electrodes.

Diaphragms of lead were placed in such a way as to reduce stray effects (chiefly lead L-radiation) to a minimum. Two were placed between the tube and the radiator, two between the radiator and the primary chamber, and two between the radiator and the secondary chamber. Diaphragms were placed also immediately in front of the windows of the ionization-chambers. All lead close to the radiator was covered with aluminium to absorb tertiary lead L-radiation. A test for stray effect showed that this was small and of the order that would be produced by scattering from the air in the region of the radiator. Filtering-sheets were placed at A (fig. 2) and absorbing-sheets at A, B, and C. The amount of radiation scattered into the chambers by the absorbing-sheets was negligible. The figure is drawn to scale.

Radiators of 268, 50, and 15 sheets of filter-paper were used. The results are shown in Table VII. and Graphs IV.

and V. The mass-absorption coefficients of Secondary and Primary are plotted as ordinates and abscissæ. Compton's theory requires an increase in wave-length on scattering of 0·0242 Å for all wave-lengths scattered at 90°. From a knowledge of relative values of wave-lengths and mass-absorption coefficients for aluminium (approximate values are good enough for the purpose), the theoretical values of $\mu/\rho_{Al}$ may be calculated for the secondary radiation. These were calculated, and the differences between the values of $\mu/\rho_{Al}$ for secondary and primary were entered alongside the experimental values in the table. Also Compton's theoretical curve is shown by the dotted line in the graphs.

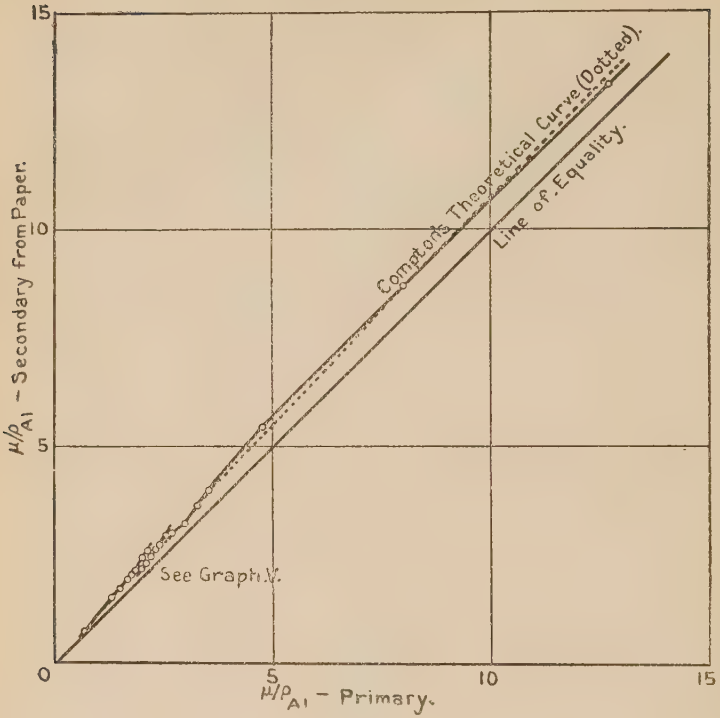
TABLE VII.

Increase ($\delta\mu/\rho$) in $\mu/\rho$ on scattering at 90° from Filter-paper.

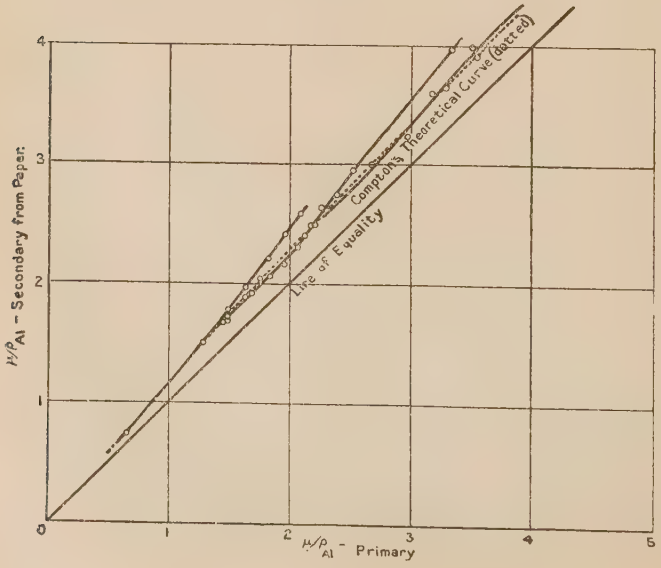
| Radiator (filter-paper). | Filter in mm. of Al. | $\mu/\rho$ Al. Primary. | $\delta\mu/\rho$. Observed. | $\delta\mu/\rho$. Theory. |
|---|---|---|---|---|
| 268 sheets. | None | 2·09 | 0·50 * | 0·27 |
| ,, | ,, | 1·49 | 0·29 | 0·21 |
| ,, | ,, | 1·49 | 0·23 | 0·21 |
| ,, | ,, | 2·27 | 0·37 | 0·28 |
| ,, | 3·66 | 0·65 | 0·092 | 0·106 |
| 50 sheets. | None | 3·55 | 0·36 | 0·38 |
| ,, | ,, | 3·29 | 0·37 | 0·35 |
| ,, | ,, | 4·78 | 0·65 | 0·47 |
| ,, | 0·4 | 3·34 | 0·63 * | 0·37 |
| ,, | ,, | 3·19 | 0·41 | 0·35 |
| ,, | 0·3 | 3·51 | 0·48 | 0·38 |
| ,, | 0·6 | 1·97 | 0·445 * | 0·26 |
| ,, | ,, | 2·31 | 0·30 | 0·29 |
| ,, | ,, | 2·40 | 0·35 | 0·29 |
| ,, | ,, | 2·20 | 0·29 | 0·28 |
| ,, | ,, | 2·13 | 0·275 | 0·27 |
| ,, | ,, | 2·18 | 0·31 | 0·28 |
| ,, | ,, | 2·08 | 0·23 | 0·27 |
| ,, | ,, | 1·96 | 0·19 | 0·26 |
| ,, | 0·92 | 1·64 | 0·325* | 0·23 |
| ,, | ,, | 1·69 | 0·23 | 0·24 |
| ,, | ,, | 1·85 | 0·21 | 0·26 |
| ,, | 1·36 | 1·30 | 0·205 | 0·19 |
| ,, | ,, | 1·48 | 0·22 | 0·21 |
| ,, | 0·5 | 2·68 | 0·33 | 0·31 |
| ,, | ,, | 2·53 | 0·425 * | 0·30 |
| ,, | ,, | 2·98 | 0·245 | 0·34 |
| ,, | 1·36 | 1·29 | 0·23 | 0·19 |
| ,, | ,, | 1·46 | 0·235 | 0·21 |
| ,, | 1·31 | 1·76 | 0·285 | 0·24 |
| ,, | ,, | 1·64 | 0·245 | 0·23 |
| ,, | ,, | 1·83 | 0·38 * | 0·26 |
| ,, | None | 8·03 | 0·67 | 0·70 |
| 15 sheets. | ,, | 12·7 | 0·69 | 0·90 |

* These observations are abnormal (see text).

Graph IV.




Graph V.



The observations on the whole are seen to agree fairly well with the theoretical values. The agreement is sufficiently close to conclude that Compton's theory gives the change in wave-length to the correct order of magnitude.

We will now show that the intensity scattered at right angles agrees with the theory. Let $\lambda$ = the wave-length of the primary, and $\lambda+\delta\lambda$ that of the secondary. According to the theory, $\delta\lambda = 0{\cdot}0242$ Å, and, if $\alpha = \delta\lambda/\lambda$, the ratio,

$$\frac{\text{Intensity of Scattered X-radiation}}{\text{Intensity of Primary radiation}},$$

is proportional to $(1+\alpha)/(1+2\alpha)^2$. The distribution of the scattered radiation is nearly the same throughout the region of wave-lengths tested, so that the ratio,

$$\frac{\text{Intensity of radiation scattered at } 90^\circ}{\text{Intensity of Primary radiation}},$$

will also be proportional to this fraction. Now let $\mu$ and $\mu+\delta\mu$ be the absorption-coefficients of the primary and secondary for the gas in the ionization-chambers. For a given intensity, the relative absorptions by the gas and therefore the relative ionizations will be given to a sufficiently close degree of accuracy by $(\mu+\delta\mu)/\mu$. The observed values of

$$\frac{\text{Ionization by Secondary}}{\text{Ionization by Primary}}$$

for a range of wave-lengths should therefore be proportional to

$$\frac{1+\alpha}{(1+2\alpha)^2}\times\frac{\mu+\delta\mu}{\mu}.$$

Now $\mu$ varies as $\lambda^3$, so that $\delta\mu/\mu = 3\delta\lambda/\lambda = 3\alpha$. It follows at once that to the first order of $\alpha$ (which is $0{\cdot}08$ at most in the present range of wave-lengths, making $\alpha^2$ about $\frac{1}{2}$ a per cent. at most) the ratio of ionizations should be proportional to unity—that is, the ratio should be constant. Within the limit of experimental error, that is to about 2 per cent., the ratio was found to be constant. We may therefore conclude that the quality and intensity of the radiation scattered at right angles to the incident radiation by filter-paper is, under the conditions of experiment described above, to a close approximation in agreement with the theory of Compton, and not with the classical theory. But the "scattered" absorption under the same conditions of experiment was also found to be in agreement with the

former theory, so that it seems very probable that the excess corpuscular radiation of paper and aluminium, observed in the previous experiment, arises from the process of scattering.

Attention should perhaps be called to the apparently discontinuous nature of the curves in Graphs IV. and V. The observations appear to lie on two or perhaps three curves lying in echelon. Where the curves overlap we get two values of the ordinate for one of the abscissa, or *vice versa*. The small circles in Graph V. (which represent single observations, and not averages of observations) are seen to lie very close to one or other of the curves drawn—so close, in fact, that in practically every case there is no doubt as to which curve each circle belongs to. As no observations have been suppressed, it would seem that the discontinuities are real, and not due to experimental error. The higher values of $\mu/\rho$ Secondary, where there are two for the same value of $\mu/\rho$ Primary, are much less numerous than the lower, so we may consider the higher value the exceptional case. The lower values are of the order calculated on Compton's theory. It should be pointed out that the higher values were not obtained consecutively. Table VII. gives the observations in the order in which they were taken, and those which definitely gave the higher or exceptional values, six in number, are marked with a star. It will be noticed that the higher values are distributed fairly uniformly throughout the series, and cannot therefore be attributed to any temporary fault in the apparatus. The probability that the effect is genuine is strengthened by the fact that other observers in this laboratory have found similar discontinuities at the same wave-lengths in performing the experiment under different conditions. While the effect may be due to the heterogeneity of the radiation, for only very thin filters were used, the possibility of a more fundamental origin should not be overlooked. When the radiation is very hard, the scattered radiation is directed strongly in the forwards direction, according to Compton's theory, and the relative intensity of the radiation at 90° is reduced. The presence of the tungsten K-lines in the incident radiation would for this reason produce less than its due effect in the secondary, and the average penetrating power of the secondary would on this account alone be less than that of the primary. This effect, combined with the normal softening of the radiation, would give too high a value of $\mu/\rho$, Secondary, on our curve. The results obtained with very heterogeneous beams, where

the exact nature of the heterogeneity is not known, would therefore appear to be difficult to interpret.

*General Discussion of Results of Experiments.*

The results of the experiments which have just been described have been shown to be in general agreement with the description of the process of scattering due to Compton. It seems impossible, however, to reconcile the results of Compton with those of Clark and Duane without assuming the existence of alternative processes of scattering. The absence of discontinuities in the absorption curves given in Graph I., when compared with the marked discontinuities in the curves of Barkla and White, would appear to require the assumption of the existence of alternative processes of absorption. It seems probable that these two phenomena are not unconnected with one another. I understand that Professor Barkla is about to publish some experimental results which may throw some light on this subject.

*Summary.*

Experiments on the relative absorptions of X-rays by copper, aluminium, and paper, under conditions detailed, show that under these conditions the J-discontinuities of Barkla and White do not appear. The results indicate that the absorption associated with scattering falls below the classical value by an amount which agrees very well with that calculated on the quantum theory of scattering of A. H. Compton.

Experiments on the ionization of sulphur dioxide and ethyl bromide give no certain indication of the existence of J-discontinuities, but they give some evidence for the existence of an abnormal corpuscular emission from air such as would be accounted for by the Compton theory.

Experiments on the corpuscular emission from paper, aluminium, copper, and lead, as measured by ionization of hydrogen, showed that the lighter elements emitted an additional corpuscular radiation when the wave-length of the incident radiation was short, that the energy required for the excess emission produced no apparent discontinuity in the absorption curve for paper, but they lent support to the view of Compton that a corpuscular emission would be found in association with scattering when the primary was of short wave-length.

An experiment on the scattering of X-rays by filter-paper showed that the change in quality and the variation in

intensity of the scattered radiation required by Compton's theory were confirmed over a considerable range of wave-lengths for scattering at right angles to the primary. From this it was deduced that the scattered X-rays were probably carrying only a fraction of the energy absorbed in association with scattering, thus strengthening the probability that the excess corpuscular emission observed in the previous experiments was to be attributed to the scattering process.

The experiments were carried out under the direction and supervision of Professor Barkla, and I take this opportunity of thanking him and expressing my appreciation of the generous amount of assistance he has given during the conduct of the experiments and in preparing the results for publication.

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## XXIV. *The Production of Sub-Harmonics.* *By* W. N. Bond, *B.A., D.Sc., F.Inst.P.* (*Lecturer in Physics, University College, Reading*) *.

[Plate IV.]

THIS paper consists of a detailed consideration of the sub-harmonic notes, produced by the intermittent contact of a tuning-fork and a suitable solid body, that have been previously described and demonstrated †. The experimental conditions will be described briefly before discussing the mechanism of the phenomenon.

When a large unmounted tuning-fork is energetically sounded, and part of the stem or the basal end of its prongs is touched lightly against a fixed solid body, some note may be produced that has a frequency equal to that of the fork divided by a small integer. As many as five different notes may thus be obtained, the smaller frequencies being produced when the amplitude of the fork is large or the force pressing the fork against the body small. The intermittent nature of the contact may be demonstrated by using a metal block for the fixed body, and completing a circuit containing an ammeter or a telephone in series with the contact and a cell. The ammeter gives different mean currents for the different sub-harmonics, and the telephone emits the corresponding notes. By using a Deprez electromagnetic signal-marker

* Communicated by the Author.

† 'Nature,' March 8th, 1924, and Proc. Phys. Soc., August 1924.

in the circuit, graphical records of the current have been obtained, and the frequencies are found to be all sub-multiples of that of the fork. This is illustrated in Pl. IV. fig. 1, where the intermittent current has frequencies equal to that of the fork (288) and its first four sub-harmonics. An intermittent current of frequency 50, also shown, was used as a standard for measurement. Experimental values of the frequencies, thus obtained, are given in the Plate.

The shape and material of the fixed body may be varied considerably, and yet the notes are obtained. But the notes are most clearly emitted when a hard metal body of high natural frequency and high elastic constants is employed (provided the metal be not such as to emit a jangle when contact is made).

Since the use of a fixed body of low natural frequency does not make the notes less difficult to obtain, it may be concluded that the intermittent nature of the contact is due to the rebound of the fork from the body (and not to a momentary depression in the surface of the body), a whole number of periods of the fork elapsing between successive contacts.

The graphical records consist of three types. Firstly, quite irregular records are found. The coarse note that these correspond to may be avoided by letting the fork approach the block very gradually. Secondly, regular records occur, such as are shown in Pl. IV. fig. 1. And thirdly, in some records the intervals between successive contacts vary, but each interval appears on casual inspection to correspond to one of the sub-harmonics (fig. 2).

If the production of the notes be supposed to be due to contact of the fork and body, a rebound of the fork, and its subsequent return because of the small applied force, various peculiarities require explanation. When the fork amplitude is large and the applied force and original velocity are small, the notes are almost invariably produced, without special adjustment of the conditions. The value of the applied force may fluctuate, and the amplitude of the fork decrease gradually, and yet as many as a hundred equal consecutive waves may occur in the case of the higher frequency notes. (Only sets of about ten consecutive waves were obtained for the fourth sub-harmonic.) And, finally, the successive intervals may differ, each being, however, approximately equal to a simple multiple of the period of the fork.

The only theory that seems to satisfy all these conditions is that the velocity of approach of the fork and block, just before an impact occurs, is small compared with the average

velocity of oscillation of the part of the fork considered (measured in the direction of the normal to the surface of contact). If this condition were satisfied at each impact, contact would invariably occur when the part of the fork was near the outward end of its oscillation; and hence any interval between impacts would only differ from a multiple of the fork's period by a small fraction of this period. It has been shown that the relative velocity of fork and body should be small at the commencement of the experiment; but for it also to be small at the end of *each* interval it must be small just after each impact, unless the motion of the fork as a whole be supposed damped. (And damping cannot in general be supposed important, as the higher notes may be obtained when the fork is erect and rests almost unheld, with its stem on the solid body.)

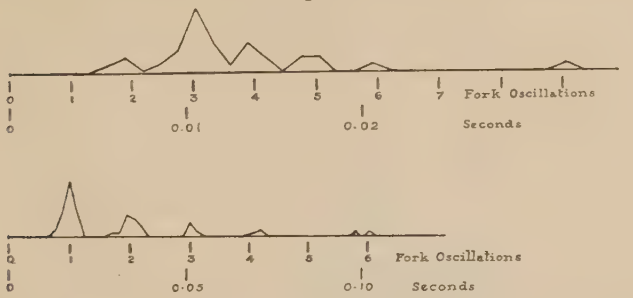
The relative velocity of the two surfaces just before contact is equal to the small velocity of approach, together with the larger velocity of oscillation of the portion of the fork making contact. That the consequent velocity of departure is small may be considered due to the contact being of the nature of an impulse. The surfaces in contact must be supposed appreciably deformed during a typical impact. The magnitude of the impulse will depend chiefly on the amplitude of the fork and the phase when contact commences; and the velocity of departure of the fork will depend also on the mass of the fork (or its moment of inertia, if it be supported by a pivot).

In order to test this hypothesis, it was decided to measure the intervals in a record of a series of impacts, where the successive intervals corresponded to various sub-harmonics. A graph obtained with the fork of frequency 288 was first used; and the positions at which contact occurred were compared with an ideal series such as would have occurred if contact had been made at the same phase at each oscillation of the fork. Owing to the necessity of any error of recording and measurement being considerably less than 1/288 of a second, the experiment was repeated using a fork of frequency $59{\cdot}1 \pm 0{\cdot}04$. With such a low note the sub-harmonics could not be heard, though they appear on the records (Pl. IV. fig. 2).

Observed intervals between consecutive impacts are represented statistically in fig. 3 from experiments with the 288 and 59·1 forks. It will be noticed that the intervals are approximately whole multiples of the period of the fork, but that deviations occur. In the case of the 288 fork, a few observed intervals differ by almost half a fork-period from

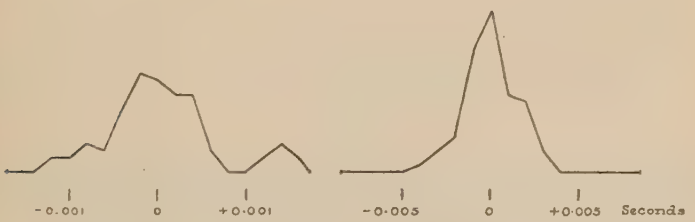
being an integral number of fork oscillations. No such large deviations were found with the slower fork, and they may therefore be supposed to be due to experimental errors.

Fig. 3.




In fig. 4 the difference, (observed time of contact — time of ideal contact) is plotted statistically for the two forks. The ideal series was chosen so that the algebraic mean of the above differences was zero for any particular series of impacts. The figure shows that, on the average, the times of contact differ from the ideal times by about a sixth of a period of the fork.

Fig. 4.




When a time of contact differs from that of the ideal set, it is found that in about 80 per cent. of the cases the next impact differs by a smaller amount or by an amount of the reverse sign (see Pl. IV. fig. 2). And when a record is almost entirely that of a single sub-harmonic (Pl. IV. fig. 1), the deviations plotted in fig. 4 become appreciably less. At the commencement of regular intermittent motion it was found that the successive periods approach more and more closely to the ideal period. These facts are explained by the theory put forward. For, a contact that is markedly earlier in phase than the extremity of the fork's oscillation will produce an abnormally large velocity of rebound ; and the succeeding contact will occur when the fork's phase is abnormally late.

The following interval will in consequence, be specially short.

No particular conditions have been noticed always to precede an interval a number of periods longer in duration than the former one (when intervals corresponding to various sub-harmonics are occurring in rapid succession). This fluctuation between various sub-harmonics may be due to change in the applied force or to slight shifting of the point of contact.

Apart from any bearing of the experiments on theories of hearing, one simple practical application may be mentioned. Provided a suitable recording arrangement is available, the frequency of any unmounted fork can be obtained by attaching a fine wire to its stem, bowing the fork, and making intermittent contact between the fork's stem and a metal block (as described above). The frequency of a simple fork, unmodified by any stile, is thus obtained, without the need of an elaborate microphonic or stroboscopic device.

In conclusion, the Author would like to thank Prof. J. A. Crowther and Mr. A. W. P. Wolters for the facilities that have made these experiments possible; and Mr. J. L. Dunk for interesting points that have arisen during correspondence on the subject.

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XXV. *Fluid Circulation Round Cylindrical Obstacles.* *By* D. M. Wrinch, *D.Sc., Lecturer at Lady Margaret Hall, Oxford* *.

Summary.

This paper treats of the motion of a fluid circulating round a fixed cylindrical obstacle in the presence of one or more line sources or sinks in any case when $c$ the curve of cross-section of the obstacle has an equation of the form

$$x = a\cos u + a_1\cos(\alpha_1 - u) + a_2\cos(\alpha_2 - 2u) + \ldots,$$
$$y = a\sin u + a_1\sin(\alpha_1 - u) + a_2\sin(\alpha_2 - 2u) + \ldots,$$

the parameters $(a_n, \alpha_n)$ being suitably restricted.

The problem of a fluid circulating round a cylindrical obstacle in the presence of a single sink is discussed in detail. The resultant pressure on the obstacle is easily found by the evaluation of the contour integral

$$\int_c (d\mathrm{W}/dz)^2 dz,$$

where

$$\mathrm{W} = \phi + i\psi,$$

and $\phi$ is the velocity potential and $\psi$ the stream-function associated with the motion.

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* Communicated by the Author.

Fig. 1.

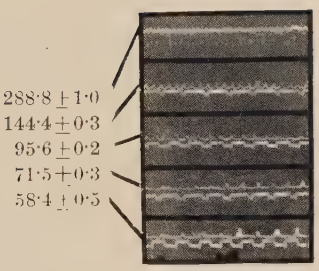


Fig. 2.

The problem when there is a source as well as a sink is also discussed. This problem, when the positions of the source and sink are suitably chosen, easily gives, as a degenerate case, the problem of fluid streaming past and circulating round a cylindrical obstacle. A similar contour integration along $c$, the curve of cross-section of the obstacle, yields the result that the force on the obstacle per unit length is $\rho kq$ in a direction $\pi/2$ from the direction of the undisturbed stream measured in the sense opposite to the sense of the circulation, where $\rho$ is the density of the fluid, $k$ the intensity of circulation, and $q$ the velocity in the undisturbed stream.

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THIS paper treats the motion of a fluid outside a fixed cylindrical obstacle in the presence of one or more line sinks or sources when there is circulation round the obstacle.

A frontal attack is made on the problem on this occasion by the discussion of the whole range of cases when the cross-section of the obstacle has an equation of the form

$$x = a\cos u + a_1\cos(\alpha_1 - u) + a_2\cos(\alpha_2 - 2u) + \ldots,$$

$$y = a\sin u + a_1\sin(\alpha_1 - u) + a_2\sin(\alpha_2 - 2u) + \ldots,$$

certain restrictions being made as to the values which may be assumed by the various parameters $(a_n, \alpha_n)$. Any number of the parameters $a_n$ may vanish, and this specification evidently covers the cases generally discussed, such as, for example, the circle, the ellipse, and the hypotrochoid.

The restrictions on the parameters $(a_n, \alpha_n)$ do not allow the inclusion of cases when the curve of cross-section of the obstacle has a node. It is possible, however, to include cases when the curve has a cusp, but in all such cases when there is a single source or sink outside the obstacle an infinite velocity is in general associated with the cusp, and the motion in each case cannot be realized with a finite pressure in the fluid. This is also the case when there is circulation about the obstacle. But if there is circulation $k$ round the obstacle in the presence of a single sink or source of strength $m$, it is possible to choose the ratio of $k$ to $m$ in such a way that the velocity at the cusp remains finite. The resulting motion is then possible with a finite pressure at infinity, provided that this pressure is sufficiently large.

In this paper the various problems are attacked in general for any value of the circulation and for sources and sinks of any strengths so long as the curve of cross-section of the obstacle has no cusps. If the curve of cross-section has a cusp, the problem is discussed only when the magnitude of

the circulation is such as to produce with the sources and sinks a finite velocity at the cusp.

The solution of the problem when there is circulation about an obstacle and a single sink is given in detail, and by an application of the theory of contour integration the resultant pressure on the obstacle per unit length is easily found. The problem when there is a line source and a line sink in addition to circulation round the obstacle is also mentioned. Finally, the problem of streaming past the obstacle round which there is circulation, which is a degenerate case of the preceding problem when the positions of the source and sink are suitably chosen, is discussed. The resultant pressure on the obstacle in this case is proved to be the product of the density of the fluid and the circulation and the velocity of the stream at infinity, and to act in a direction $\frac{\pi}{2}$ from the direction of the stream at infinity measured in the sense opposite to that of circulation. This result, known as the Kutta-Joukowski Lift Formula, is proved by Joukowski * by general dynamical principles. The alternative proof given in the present paper proceeds by means of a simple application of the theory of contour integration which allows the rapid calculation of the resultant pressure on an obstacle due to any motion. It is deemed worthy of inclusion in this paper because of the great importance of the Lift Formula in the Circulation Theory of Lift † now under discussion in Aerodynamics.

## I. *A line sink in the presence of a cylindrical obstacle.*

Suppose that there is a line sink of strength $m$ at a point $z_0$ outside a cylinder. The line sink alone in space has the W-function $W_0$ given by

$$W_0=\phi_0+i\psi_0=\frac{m}{2\pi}\log(z_0-z),$$

where $\phi_0$ is the velocity potential and $\psi_0$ the stream-function associated with the motion. Let $W_1$ represent the disturbance due to the presence of the cylinder, so that the

* N. Joukowski, *Aérodynamique* (*traduit du russe par* S. Drzewiecki), Paris, 1916, chap. vi.

† See F. W. Lanchester, 'Aerodynamics,' ch. iv.; A. R. Low, 'The Circulation Theory of Lift,' International Air Congress, London, 1923; 'Dictionary of Applied Physics,' vol. v. 1923; 'The Hydrodynamical Theory of Wing-surfaces,' pp. 217–220; and other text-books.

motion when the sink is in the presence of the cylinder is given by

$$W=\phi+i\psi=W_0+W_1=\phi_0+\phi_1+i(\psi_0+\psi_1).$$

Then we require that $W_1$ shall be evanescent on the circle at infinity, and that it shall satisfy Laplace's equation and have no singularities between the circle at infinity and $c$ the curve of cross-section of the cylinder, and further that $\psi$ shall be constant on $c$.

Now consider the transformation

$$z=ae^{iw}+a_1e^{i\alpha_1-iw}+a_2e^{i\alpha_2-2iw}+\ldots, \quad . \quad . \quad . \quad (1{\cdot}1)$$

where

$$w=u+iv,$$

which has the property that $v=-\infty$ on the circle at infinity. Then $v=0$ on the curve whose equation is given by

$$z=ae^{iu}+a_1e^{i\alpha_1-iu}+a_2e^{i\alpha_2-2iu}+\ldots.,$$

or

$$\left.\begin{aligned} x&=a\cos u+a_1\cos(\alpha_1-u)+a_2\cos(\alpha_2-2u)+\ldots.,\\ y&=a\sin u+a_1\sin(\alpha_1-u)+a_2\sin(\alpha_2-2u)+\ldots.\end{aligned}\right\} \quad (1{\cdot}2)$$

Suppose, further, that the constants in the transformation (1) are such that for values of $v$ between zero and $-\infty$, $dz/dw$ has no zeros and no infinities. Then we may at once solve the problem of the sink outside a cylindrical obstacle in all cases where, for some value or other of the various constants satisfying the condition with respect to the zeros and infinities of $dz/dw$, the equation of $c$, the curve of cross-section of the cylinder, is given by (1·2).

For, in terms of the complex variable $w$, we have

$$\begin{aligned} W_0&=\frac{m}{2\pi}\log\{ae^{iw_0}+a_1e^{i\alpha_1-iw_0}+a_2e^{i\alpha_2-2iw_0}+\ldots\\ &\qquad -(ae^{iw}+a_1e^{i\alpha_1-iw}+a_2e^{i\alpha_2-2iw}+\ldots)\}\\ &=\frac{m}{2\pi}\log(e^{iw_0}-e^{iw})\{a-a_1e^{i\alpha_1-iw_0-iw}\\ &\qquad -a_2e^{i\alpha_2}(e^{-iw-2iw_0}-e^{-iw_0-2iw})-\ldots\},\end{aligned}$$

$w_0$ being given by

$$z_0=ae^{iw_0}+a_1e^{i\alpha_1-iw_0}+a_2e^{i\alpha_2-2iw_0}+\ldots.$$

Since $z_0$ lies in the region between $c$ and the circle at infinity, $v_0$ lies between zero and $-\infty$. We may evidently

expand $W_0$ in a series convergent in the neighbourhood of the curve $c$ in the form

$$W_0 = \frac{miw_0}{2\pi} - \frac{m}{2\pi}(e^{iw-iw_0} + \tfrac{1}{2}e^{2iw-2iw_0} + \ldots) + \frac{m}{2\pi}\log(a - a_1 e^{-iw_0-iw}\ldots).$$

Then, if we write

$$W_1 = -\frac{m}{2\pi}(e^{v_0}e^{-iw+iu_0} + \tfrac{1}{2}e^{2v_0}e^{-2iw+2iu_0} + \ldots) - \frac{m}{2\pi}\{\log(a - a_1 e^{-iw_0-iw}\ldots) - \log a\},$$

$W_1$ is evanescent on the circle at infinity and, further

$$W = \frac{miw_0}{2\pi} - \frac{m}{\pi}\{e^{v_0}\cos(w-u_0) + \tfrac{1}{2}e^{2v_0}\cos 2(w-u_0) + \ldots\}.$$

With this determination of W we have

$$\phi = -\frac{mv_0}{2\pi} - \frac{m}{\pi}\{e^{v_0}\cos(u-u_0)\cosh v + \tfrac{1}{2}e^{2v_0}\cos 2(u-u_0)\cosh 2v + \ldots\},$$

$$\psi = \frac{mu_0}{2\pi} + \frac{m}{\pi}\{e^{v_0}\sin(u-u_0)\sinh v + \tfrac{1}{2}e^{2v_0}\sin 2(u-u_0)\sinh 2v + \ldots\},$$

and on the curve $c$, $\psi$ is constant and equal to $\frac{mu_0}{2\pi}$. We therefore have the W-function for the motion when the sink $m$ at $z_0$ is in the presence of the cylindrical obstacle whose cross-section is the curve $c$ given by (2).

## II. *Circulation about a cylinder in the presence of a line sink.*

We may also solve the more general problem of a circulation $-k$ about this cylinder in the direction $u$ increasing when there is a line sink $m$ at the point $z_0$. We evidently have

$$W = \frac{kw}{2\pi} + \frac{miw_0}{2\pi} - \frac{m}{\pi}\{e^{v_0}\cos(w-u_0) + \tfrac{1}{2}e^{2v_0}\cos 2(w-u_0)\ldots\}.$$

### III. *The pressure on the cylinder.*

It is a simple matter to find the resultant pressure on the obstacle in the presence of the sink, and also in the more general case when there is circulation round the obstacle in addition. For let $q$ represent the velocity at any point and $\phi$ the angle which the velocity makes with the axis of $x$ in the case of the sink $m$ and circulation $-k$. Then

$$qe^{-i\phi} = dW/dz = \frac{k}{2\pi} + \frac{m}{\pi}\{e^{v_0}\sin(w-u_0) + e^{2v_0}\sin 2(w-u_0) + \ldots\}$$

$$= k/2\pi + \frac{m}{2\pi}\frac{e^{v_0}\sin(w-u_0)}{\cosh v_0 - \cos(w-u_0)}.$$

The pressure at any point is given by

$$p = -\tfrac{1}{2}\rho q^2 + \text{constant}.$$

We will discuss only those motions in which $q$ is finite everywhere except at the sink, for these motions alone are physically possible. Now $dW/dw$ is infinite only at the sink. Further, $dz/dw$ is never zero between the curve $c$ and the circle at infinity. On the curve $c$, however, we can admit a simple zero of $dz/dw$ and consequently a cusp, provided that the ratio of $k/m$ is so chosen that $dW/dw$ has a zero at this point. Thus suppose that $dz/dw$ has a simple zero at the point $u=v=0$ on the curve $c$. If we choose the ratio $k/m$ in such a way that $dW/dz$ vanishes at this point, so that, in fact,

$$me^{v_0}\sin u_0 = k(\cosh v_0 - \cos u_0), \quad . \quad . \quad . \quad (3.1)$$

the velocity is still everywhere finite. We therefore deal with the general case of a sink $m$ and circulation $-k$ when the curve $c$ has no cusps and with the case when the curve has a cusp or cusps only when the ratio $k/m$ can be and is so adjusted as to make the velocity at the cusp or cusps remain finite.

Let $p_x$ and $p_y$ be the resultant pressures in the directions $x$ and $y$. Then

$$p_y + ip_x = \int p(dx - i\,dy)$$

$$= -\tfrac{1}{2}\rho\int q^2(dx - i\,dy),$$

the integrations being taken along the curve. But on the curve $c$ the velocity is wholly tangential to the curve.

Therefore on $c$

$$q^2(dx-i\,dy)=(d\mathrm{W}/dz)^2(dx/ds+i\,dy/ds)^2(dx-i\,dy)$$
$$=(d\mathrm{W}/dz)^2(dx+i\,dy).$$

Thus

$$p_y+ip_x=\tfrac{1}{2}\rho\int_c q^2(dx-i\,dy)=\tfrac{1}{2}\rho\int_c(d\mathrm{W}/dz)^2dz^*.$$

To integrate $(d\mathrm{W}/dz)^2$ round the curve $c$ in the $z$-plane, we use the transformation

$$z=ae^{iw}+a_1e^{i\alpha_1-iw}+a_2e^{i\alpha_2-2iw}+\ldots,$$

and write

$$t=e^{-iw}.$$

Then the integration of $(d\mathrm{W}/dz)^2$ round the curve $c$ becomes the integration round the unit circle $t=e^{i\theta}$ in the $t$-plane from $\theta=\pi$ to $\theta=-\pi$ of

$$(d\mathrm{W}/dz)^2dz/dw\,.\,dw/dt=(d\mathrm{W}/dw)^2.\,(dw/dt)/(dz/dw).$$

Now

$$d\mathrm{W}/dw=\frac{m}{2\pi}\cdot\frac{\sin(w-u_0)}{\cosh v_0-\cos(w-u_0)}+\frac{k}{2\pi}$$
$$=-\frac{mi}{2\pi}\frac{t^2e^{v_0}-t_0{}^2e^{-v_0}}{(t-t_0)(te^{v_0}-t_0e^{-v_0}}+\frac{k}{2\pi},$$

and

$$(dw/dt)/(dz/dw)=1/(a-a_1t^2e^{i\alpha_1}-2a_2t^3e^{i\alpha_2}\ldots).$$

There is evidently no pole of the integrand at $t=0$ and only a pole therefore at $t=t_0=e^{-iw_0}$. There is no pole on the contour of integration corresponding to a point on the curve $c$, for in the cases when there is a zero of $dz/dw$ at such a point the ratio of $k/m$ is so chosen as to produce a zero in $d\mathrm{W}/dw$ at the same point.

Now putting $t=t_0+\epsilon$, we have

$$(d\mathrm{W}/dw)^2(dw/dt)/(dz/dw)$$
$$=\left[\frac{k}{2\pi}-\frac{mit_0}{\pi e}\cdot\frac{t_0\sinh v_0+\epsilon e^{v_0}\ldots}{2t_0\sinh v_0+\epsilon e^{v_0}\ldots}\right]^2\Big/a\mathrm{A}(1-\lambda\epsilon\ldots)$$
$$=\frac{1}{a\mathrm{A}\epsilon^2}\left\{\frac{k\epsilon}{2\pi}-\frac{mit_0}{2\pi}\left(1+\frac{\epsilon e^{v_0}}{2t_0\sinh v_0}\right)\right\}^2(1+\lambda\epsilon)+\ldots,$$

provided that

$$a-a_1t^2e^{i\alpha_1}-2a_2t^3e^{i\alpha_2}\ldots=a\mathrm{A}(1-\lambda\epsilon+\ldots).$$

* This theorem is sometimes referred to as Blasius's Theorem.

Thus the residue at $t_0$ is given by

$$\frac{1}{Aa}\left\{\left(\frac{mit_0}{2\pi}\right)^2\lambda-\frac{mit_0}{\pi}\left(\frac{k}{2\pi}-\frac{mie^{v_0}}{4\pi\sinh v_0}\right)\right\}.$$

The value of the integral is therefore

$$\frac{2\pi i}{aA}\cdot\frac{mt_0}{2\pi}\left\{\frac{mt_0\lambda}{2\pi}+\frac{ik}{\pi}+\frac{me^{v_0}}{2\pi\sinh v_0}\right\}$$

$$=\frac{mie^{-iw_0}}{2\pi aA}\left\{\frac{me^{v_0}}{\sinh v_0}+me^{-iw_0}\lambda+2ik\right\},$$

the integration along the unit circle $|t|=1$ being taken in the clockwise direction. Therefore we have

$$-p_y-ip_x=\tfrac{1}{2}\rho\int_c(d\mathrm{W}/dz)^2dz$$

$$=\frac{m\rho}{4\pi aA}\left\{\frac{me^{v_0}}{\sinh v_0}u^{-iw_0}-2ke^{-iw_0}+m\lambda ie^{-2iw_0}\right\}.$$

It is of particular interest to notice the presence of the force due to the combination of circulation with the sink. This "lift" acts in a direction determined by the orientation of the sink with respect to the obstacle and by the argument of A, which also depends on this orientation and on the equation of $c$, the curve of cross-section. It is also of interest to notice the critical value, given in (3.1), of the ratio $k/m$ requisite in the case of a cusped curve if an infinite velocity at the cusp is to be avoided.

The case when the curve $c$ is symmetric about the axis of $x$ is of special interest. We then have $\alpha_1=\alpha_2\ldots=0$, and a simpler form results. If, in addition, we are concerned only with a sink on this axis of symmetry on the positive side $u_0=0$, and with A and $\lambda$ wholly real, we have

$$p_y=m\rho ke^{v_0}/2\pi aA,$$

$$p_x=-\frac{m^2\rho e^{2v_0}}{4\pi aA}\left(\frac{1}{\sinh v_0}+\lambda\right).$$

Thus the sink $m$ exerts a force proportional to the product $mk$ in a direction $\pi/2$ from the direction of the $x$-axis measured in the sense opposite to that of the circulation $k$. It also exerts a force proportional to $m$ and independent of $k$ along the $x$-axis. It therefore appears that the lift in this case is perpendicular to the line joining the sink to the origin of coordinates.

IV. *The pressure on a circular cylinder.*

Thus, to take a simple example of this result: if $c$ is a circle of radius $a$, $A=1$, $\lambda=0$, and a sink at a point $f$ from the centre on the positive side of the $x$-axis together with circulation $k$ clockwise yields the forces

$$p_y=m\rho k/4\pi f,$$
$$p_x=m^2\rho a^2/2\pi f(f^2-a^2).$$

Thus the sink attracts the cylinder with a force

$$m^2\rho a/2\pi f(f^2-a^2)$$

per unit length along the line joining the centre to the sink, and provides also a transverse lifting force

$$m\rho k/4\pi f$$

per unit length in a direction $\pi/2$ from this line in the counter-clockwise direction. And it is to be remarked that a source $m$ would in the same way attract the cylinder along the line joining it to the centre of the cross-section; the transverse force in this case would, however, be in a direction making an angle $\pi/2$ measured clockwise with this line.

V. *The pressure on an elliptic cylinder.*

As another simple illustration we may consider the forces when a line sink $m$ is situated at a point $f$ on the $x$-axis outside the elliptic cylinder

$$x=\alpha\cos u,$$
$$y=\beta\sin u.$$

Here we have the transformation

$$z=ae^{iw}+a_1e^{-iw}$$

with

$$a=\tfrac{1}{2}(\alpha+\beta), \qquad b=\tfrac{1}{2}(\alpha-b).$$

Further, $v_0=0$ and

$$f=ae^{-v_0}+a_1e^{v_0},$$

so that

$$e^{v_0}=(f-\sqrt{f^2-4aa_1})/2a_1=(f-\sqrt{f^2-c^2})/(\alpha-\beta),$$

where

$$e^2=\alpha^2-\beta^2$$

and

$$e^{-v_0}=(f+\sqrt{f^2-4aa_1})/2a=(f+\sqrt{f^2-c^2})/(\alpha+\beta),$$

so that

$$\sinh v_0=(f-\sqrt{f^2-c^2})/2(\alpha-\beta)-(f+\sqrt{f^2-c^2})/2(\alpha+\beta)$$
$$=(f\beta-\alpha\sqrt{f^2-c^2})/c^2.$$

Also
$$dz/dw = \frac{ai}{t}\left(1-\frac{a_1}{a}t^2\right).$$

Therefore
$$aA = a\left(1-\frac{a_1}{a}t_0^2\right)$$
$$= a\left(1-\frac{a_1}{a}e^{2v_0}\right)$$
$$= e^{v_0}\sqrt{f'^2-c^2},$$

and
$$aA\lambda - 2a_1t_0 = (\alpha-\beta)e^{v_0}.$$

Thus
$$p_y = kme/4\pi(f'^2-c^2)^{1/2},$$

$$p_x = \frac{m^2\rho e^{v_0}}{4\pi(f'^2-c^2)^{1/2}}\left\{\frac{\alpha^2-\beta^2}{\alpha(f'^2-c^2)^{1/2}-\beta f'} - \frac{\alpha-\beta}{(f'^2-c^2)^{1/2}}\right\}$$
$$= \frac{m^2\rho[f'-(f'^2-c^2)^{1/2}]\beta[f'+(f'^2-c^2)^{1/2}]}{4\pi[\alpha(f'^2-c^2)^{1/2}-pf'](f'^2-c^2)}$$
$$= \frac{m^2\rho\beta c^2}{4\pi(f'^2-c^2)\{\alpha(f'^2-c^2)^{1/2}-\beta f'\}}.$$

VI. *Circulation about a cylinder in the presence of a number of line sources and sinks.*

We may, after the same manner, find the motion when there is circulation round the cylinder and any number of line sinks or sources outside. Thus, if there is a source $m$ at a point $z_0'$ in addition to a sink $m$ at a point $z_0$ where

$$z_0 = ae^{iw_0} + a_1e^{ia_1-iw_0} + a_2e^{ia_2-2iw_0} \ldots.$$
$$z_0' = ae^{iw_0'} + a_1e^{ia_1-iw_0'} + a_2e^{ia_2-2iw_0'},$$

the motion is given by

$$W = \frac{kw}{2\pi} + \frac{mi}{2\pi}(w_0-w_0') - \frac{m}{\pi}\{e^{v_0}\cos(w-u_0) - e^{v_0'}\cos(w-u_0') + \ldots.\}.$$

In particular, if $w_0' = w_0+\pi$, so that

$$v_0' = v_0, \qquad u_0' = u_0+\pi,$$

and

$$W = \frac{kw}{2\pi} + \frac{mi}{2} - \frac{2m}{\pi}\{e^{v_0}\cos(w-u_0) + \tfrac{1}{3}e^{3v_0}\cos(w-u_0) + \ldots\}.$$

If, now, we make $m \to \infty$ and $|z_0| \to \infty$ in such a way that

$$m/\pi z_0 = q_0,$$

we get streaming past the cylinder, the velocity of the stream before it is disturbed by the presence of the cylinder being $q_0$ in a direction making an angle $u_0$ with the axis of $x$, and

$$W = \frac{kw}{2\pi} - 2aq_0 \cos(w - u_0)$$

and

$$\frac{dW}{dw} = \frac{k}{2\pi} + 2aq_0 \sin(w - u_0).$$

Thus, in the case of a curve of cross-section for which $dz/du = 0$ at $u = u_1$, we need to take

$$k = 4\pi a q_0 \sin(u_0 - u_1)$$

if an infinite velocity at the cusp is to be avoided.

The value of $p_x$ and $p_y$ can be found at once. We have

$$\int_c (dW/dz)^2 dz$$

$$= \int_{c'} \left\{ \left[ \frac{k}{2\pi} + iaq_0 \left( te^{iu_0} - \frac{1}{t} e^{-iu_0} \right) \right]^2 \Big/ \left[ a - a_1 t^2 e^{ia_1} \ldots . \right] \right\} dt,$$

the latter integration being taken round $c'$ the unit circle $|t| = 1$ in a clockwise direction. The only pole is the double pole at the origin: there is, as before, no pole on the contour of integration even if $dz/dw$ has a zero at such a point, for under those circumstances the ratio $k/q_0$ is so chosen as to produce a zero in $dW/dw$ at that point. Thus

$$\int_c (dW/dz)^2 dz = -2q_0 k e^{-iu_0},$$

and

$$p_x = -\rho q_0 k \sin u_0,$$

$$p_y = \rho q_0 k \cos u_0.$$

Thus the resultant force is $\rho q_0 k$ in a direction making $\pi/2$ with the direction of the undisturbed stream measured in the sense opposite to that of circulation. This is the Kutta-Joukowski result *, which is of importance in Aerodynamics.

* See Joukowski, *loc. cit.*; also Lamb, 'Hydrodynamics,' 5th edition, pp. 76, 647.

# XXVI. *The J Transformation of Scattered X-Rays.* By C. G. Barkla, *F.R.S., and* S. R. Khastgir, *Ph.D., University of Edinburgh* *.

SINCE the earliest experiments on the scattering of X-rays it has been known that certain condtions were necessary in order to obtain *evidence* of the purest scattering of X-radiation. Under these conditions, through a wide range of wave-lengths, the secondary radiation did not differ appreciably in penetrating power from the primary. It was *this* secondary radiation, and no other, which was called the scattered X-radiation ; it was this which we showed could be explained in its polarization, in its distribution, in its intensity, precisely by the application of the classical theory as given by J. J. Thomson. (It gave exactly the number of extra-nuclear electrons per atom.) In the experimental investigations it was observed or arranged that the complications appearing under other conditions were not introduced.

Under other conditions—conditions which appear to have been obtained by many experimenters on the subject—there was a marked difference between the primary and secondary radiation *as observed.* (See Phil. Mag. April 1923 : Barkla and Mrs. Sale.) Whatever the explanation—there were several possibilities—this secondary radiation was obviously something different from the scattered radiation. This, however, has *also* recently been called by Compton the scattered radiation, and the classical theory of scattering has then been described as inadequate. As the classical theory of scattering was intended only to explain scattering, it is not surprising that it does not cover the other phenomenon which has been given the same name.

This phenomenon has been briefly discussed by one of us in a letter to 'Nature' †, and more fully in a lecture to the Röntgen Society ‡, and its application to the scattering of X-rays has been indicated. A fuller treatment of the subject will appear in other papers, but we propose to give a short description of one only of a series of experiments showing how the true nature of the phenomenon of scattering has in many cases been obscured.

Preliminary results of the kind described in this paper

* Communicated by the Authors.

† November 17, 1923.

‡ A summary only has been published in 'Nature,' November 22, 1924. A more detailed account will be given in the Journal of the Röntgen Society.

were obtained by Barkla and Mrs. Sale several years ago (1921–22); but they were so new, so puzzling, that they were withheld from publication for more careful and critical examination. The results were these:—When primary and secondary rays were compared by their power of penetrating thin absorbing sheets, they appeared practically identical; but thicker absorbing sheets showed a most decided difference (about 12 per cent. in absorption coefficient).

Again, it was found necessary to use thin radiators to obtain scattered rays like the primary throughout a long range of wave-lengths. This is indicated on p. 747*. "When thin sheets of paper were used as the scattering substance—in which case trouble due to differential absorption of primary and secondary beams in the radiating substance was avoided—experiments showed that the scattered was identical with the primary radiation throughout almost the whole range of wave-lengths experimented upon." The complication evidently introduced by thick radiating sheets was purposely avoided in order to obtain the conditions most favourable to the study of fundamental processes.

The experiment described below showed, as in many previous experiments, that the secondary radiation from paper was originally exactly like the primary radiation in penetrating power. When secondary and primary radiations were transmitted through equal thin sheets of aluminium, this equality of penetrating powers persisted only up to a certain critical thickness of absorbing material. An abrupt absorption of the secondary radiation then occurred, and the transmitted radiation was subsequently more absorbable than the primary radiation which was transmitted through exactly the same thickness of aluminium.

Thus the scattered X-radiation was modified by the J phenomenon quite outside the scattering substance, and afterwards appeared as what A. H. Compton regards (presumably) as the quantum scattered radiation. As we have said previously, the phenomenon here has no connexion with the *process* of scattering. Its association with the scattered radiation is quite accidental,—in the sense that it is not fundamental,—for an exactly similar phenomenon may be shown with a primary radiation.

*Experiment.*

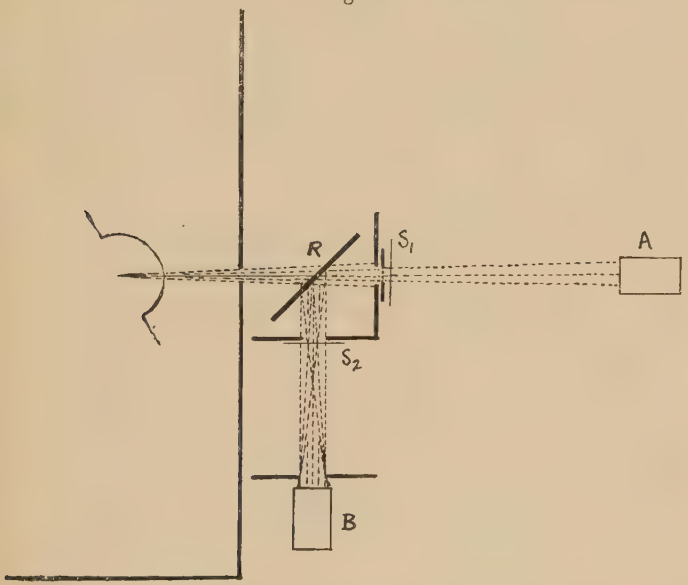
The experiment was quite a simple one. A and B were two ionization chambers—one, A, receiving pencils of

* Phil. Mag. April 1923.

primary radiation ; B receiving radiation scattered from the paper sheets R (fig. 1).

The ratio $\frac{\text{Ionization in B}}{\text{Ionization in A}}$ was determined. Thin similar sheets of aluminium were placed at $S_1$ and $S_2$, and the above ratio was observed for each thickness of aluminium.

Fig. 1.




The relation between thickness of aluminium and this ratio is shown in Table I. and is represented graphically in fig. 2.

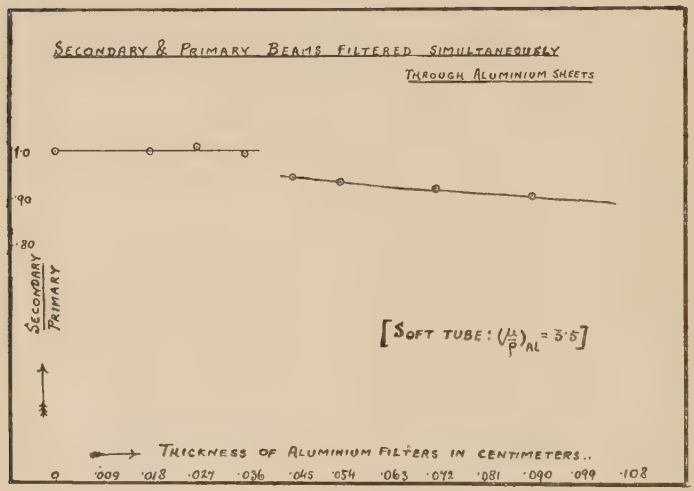
TABLE I.

Soft tube : $\left(\frac{\mu}{\rho}\right)_{Al} = 3{\cdot}5$ (from 50 per cent. absorption).

| Thickness of Al filters. | Observed ratio $\frac{\text{Secondary}}{\text{Primary}}$ (in arbitrary units). |
|---|---|
| 0·0 cm. | 1·00 × $x$ |
| ·018 ,, | 1·00 ,, |
| ·027 ,, | 1·01 ,, |
| ·036 ,, | ·99(5) ,, |
| ..... | ..... |
| ·045 ,, | ·94(5) ,, |
| ·054 ,, | ·93(5) ,, |
| ·072 ,, | ·92 ,, |
| ·090 ,, | ·90(5) ,, |

With 2, 3, and 4 sheets the ratio remained unchanged, showing a very exact equality in penetrating powers and thus in wave-lengths of the primary and secondary radiations. The fifth sheet (giving a total thickness of ·45 millimetre) produced a very big absorption of the secondary beam, as shown by the ratio suddenly dropping from 1·00 to 0·94(5). Subsequent sheets absorbed the secondary beam more than the primary. This is shown by the steady diminution of the ratio for 5, 6, 8, and 10 sheets. In the graph of fig. 2 the first horizontal portion shows equality of penetrating powers of the secondary and primary; the sudden drop shows the large

Fig. 2.


SECONDARY & PRIMARY BEAMS FILTERED SIMULTANEOUSLY
THROUGH ALUMINIUM SHEETS
1·0
·90
·80
SECONDARY / PRIMARY
[SOFT TUBE: $(\frac{\mu}{\rho})_{Al} = 3·6$]
THICKNESS OF ALUMINIUM FILTERS IN CENTIMETERS..
0 ·009 ·018 ·027 ·036 ·045 ·054 ·063 ·072 ·081 ·090 ·099 ·108

absorption of energy of the secondary beam, and the slope of the curve to the right shows the transformation which has been produced by transmission through the aluminium. The particular sheet at which the transformation occurs depends upon the original penetrating power of the radiation. With harder radiations the discontinuity is displaced to the left of the figure (see fig. 3 and Table II.), and with softer radiations to the right; for the transformation seems to occur when a certain average penetrating power is reached—in this case by filtering. Indeed, with a more penetrating radiation the discontinuity is lost sight of, for even the first thin sheet of absorbing material shows that the difference between the primary and secondary

radiations has already been produced. Under such conditions the true nature of the phenomenon is concealed.

Fig. 3.

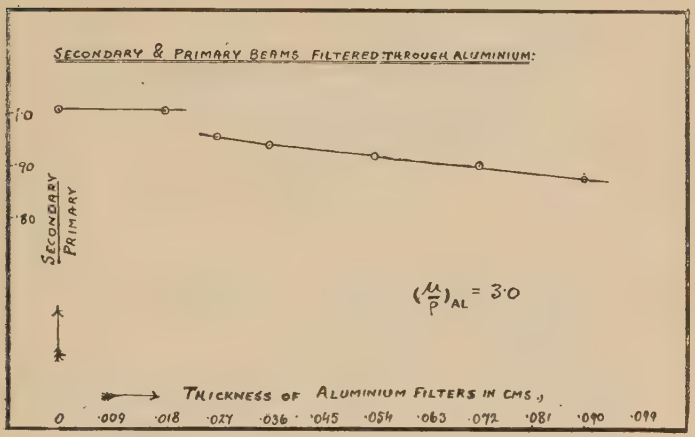



TABLE II.

Soft tube: $\left(\frac{\mu}{\rho}\right)_{Al} = 3$ (from 50 per cent. absorption).

| Thickness of Al filters. | Observed ratio $\frac{\text{Secondary}}{\text{Primary}}$ (in arbitrary units). |
|---|---|
| 0·0 cm. | 1·01 $\times x$ |
| ·018 „ | 1·00(5) „ |
| ..... | ..... |
| ·027 „ | ·95(5) „ |
| ·036 „ | ·94 „ |
| ·054 „ | ·92 „ |
| ·072 „ | ·90 „ |
| ·090 „ | ·87(5) „ |

Similar discontinuities have been obtained by filtering the radiations both by paper and by copper. Fig. 4, plotted from the results of Table III., show the corresponding phenomenon in paper. Experiments show that it is necessary to cut off a little more by aluminium than by paper to give the discontinuity—a result perfectly consistent with the conclusions from experiments of a different kind.

Fig. 4.

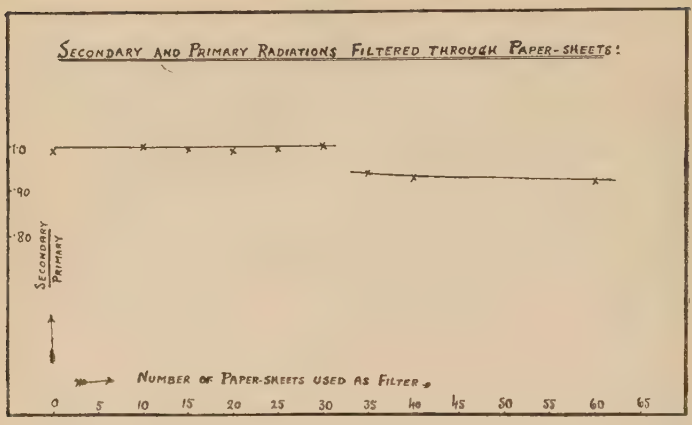



TABLE III.

Soft tube : $\left(\frac{\mu}{\rho}\right)_{Al} = 3{\cdot}5$ (approx. from 50 per cent. absorption).

| Number of Paper-sheets. | $\frac{\text{Secondary}}{\text{Primary}}$ (in arbitrary units). |
|---|---|
| 0 | ·99 × *x* |
| 10 | 1·00 ,, |
| 15 | ·99(5) ,, |
| 20 | ·99 ,, |
| 25 | ·99(5) ,, |
| 30 | 1·00 ,, |
| ..... | ..... |
| 35 | ·94 ,, |
| 40 | ·93 ,, |
| 60 | · 92 ,, |

These are only a few of a very large number of experimental results illustrating the "J" phenomenon. This has been briefly described elsewhere, and is receiving fuller treatment in a series of papers. In the case described above, it was the secondary radiation which exhibited the J phenomenon.

Our present purpose in taking this experimental result from its context is to show that the transformation in the secondary radiation has not been produced in the process of scattering. Scattering followed the classical laws; the J transformation gave the appearance of a change of wave-length in scattering.

As previously stated, the J process is probably governed by quantum laws—but it is quite distinct and fundamentally different in origin and nature from the process of scattering.

## XXVII. *A Quantitative Analysis of a Four Carrier, Voss Machine.* *By* Alfred W. Simon *.

A GENERAL quantitative theory of the influence electrostatic generator has recently been outlined by the author †.

The present work is concerned with the detailed analysis of a very simple electrostatic generator, one which may be taken as the prototype of the Voss or Toepler-Holtz machines. It consists (fig. 1) of six elements: four carriers and two inductors, and operates as follows:—

Fig. 1.

When the elements assume the configuration shown, two of the carriers, namely those in positions 4 and 6, are earthed; while two others, namely those in positions 3 and 5, are connected to the inductors 1, 2. A quarter turn later the two which were previously earthed are connected to the inductors, while the two which were previously connected to the inductors are now earthed.

It is assumed that the carriers are exactly alike and symmetrically mounted, so that a quarter turn brings the whole system again into the same geometrical configuration. In the mathematical treatment of such a machine the individual carriers lose their identity, and it is necessary to consider the electrical coefficients of the elements in only one

* Communicated by Prof. A. H. Compton.

† Proc. Nat. Acad. of Sciences, x. p. 302 (1924).

configuration (that of the figure), irrespective of which carrier, in particular, occupies any one position.

This fact is reflected in our notation, which may now be defined. We shall let $q_{1n}$ represent the charge of the inductor 1 at the beginning of the $n$th cycle (quarter turn) ; $q_{2n}$, the charge of the inductor 2 at the beginning of the $n$th cycle ; $q_{3n}$ the charge on the carrier *which occupies position* 3 at the beginning of the $n$th cycle ; and so on. Further, we shall let $V_{1n}$ and $V_{2n}$ represent the potentials of the inductors 1 and 2 respectively at the beginning of the $n$th cycle ; $V_{1n+1}$ and $V_{2n+1}$, the potentials of the inductors 1 and 2 respectively at the beginning of the $(n+1)$st cycle ; and so on.

We then have, according to well-known electrostatic theory, for the $n$th cycle :—

$$\left.\begin{aligned} q_{1n} &= a_{11}V_{1n} + a_{21}V_{2n} + a_{31}V_{1n} + a_{51}V_{2n}, \\ q_{2n} &= a_{12}V_{1n} + a_{22}V_{2n} + a_{32}V_{1n} + a_{52}V_{2n}, \\ q_{3n} &= a_{13}V_{1n} + a_{23}V_{2n} + a_{33}V_{1n} + a_{53}V_{2n}, \\ q_{4n} &= a_{14}V_{1n} + a_{24}V_{2n} + a_{34}V_{1n} + a_{54}V_{2n}, \\ q_{5n} &= a_{15}V_{1n} + a_{25}V_{2n} + a_{35}V_{1n} + a_{55}V_{2n}, \\ q_{6n} &= a_{16}V_{1n} + a_{26}V_{2n} + a_{36}V_{1n} + a_{56}V_{2n}, \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

and for the $(n\text{th}+1)$st cycle :

$$\left.\begin{aligned} q_{1n+1} &= a_{11}V_{1n+1} + a_{21}V_{2n+1} + a_{31}V_{1n+1} + a_{51}V_{2n+1}, \\ q_{2n+1} &= a_{12}V_{1n+1} + a_{22}V_{2n+1} + a_{32}V_{1n+1} + a_{52}V_{2n+1}, \\ q_{3n+1} &= a_{13}V_{1+n1} + a_{23}V_{2n+1} + a_{33}V_{1n+1} + a_{53}V_{2n+1}, \\ q_{5n+1} &= a_{15}V_{1+n1} + a_{25}V_{2n+1} + a_{35}V_{1n+1} + a_{55}V_{2n+1}, \end{aligned}\right\} \quad . \quad (2)$$

Further, if we assume no leakage, we can put

$$\left.\begin{aligned} q_{1n+1} + q_{3n+1} &= q_{1n} + q_{6n}, \\ q_{2n+1} + q_{5n+1} &= q_{2n} + q_{4n}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

In the system of equations (1), (2), (3) there are 12 equations and 14 unknowns : hence by elimination we can derive two simultaneous equations of the form

$$V_{1n+1} = LV_{1n} - MV_{2n}, \quad . \quad . \quad . \quad . \quad (4\,a)$$

$$V_{2n+1} = SV_{2n} - RV_{1n}, \quad . \quad . \quad . \quad . \quad (4\,b)$$

where L, M, R, S are certain functions of the coefficients of induction and coefficients of capacity involved in equations (1).

If we solve $(4b)$ for $V_{2n}$, we have

$$MV_{2n}=LV_{1n}-V_{1n+1},$$

and, raising the subscripts by unity, we have, further,

$$MV_{2n+1}=LV_{1n+1}-V_{1n+2}.$$

Substituting these values in $(4a)$, we obtain

$$V_{1n+2}-KV_{1n+1}+NV_{1n}=0, \quad . \quad . \quad . \quad (5a)$$

where

$$K=(L+S),$$

$$N=(LS-RM),$$

Similarly, if we eliminate the terms of the $V_1$ sequence, we obtain

$$V_{2n+2}-KV_{2n+1}+NV_{2n}=0 \quad . \quad . \quad . \quad . \quad (5b)$$

Now, the theory of finite differences * applied to $(5a)$ gives for $V_{1n}$ an equation of the form

$$V_{1n}=C_1r_1^n+C_2r_2^n, \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

and applied to $(5b)$, for $V_{2n}$

$$V_{2n}=C_3r_1^n+C_4r_2^n, \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

where $r_1$, $r_2$ are the roots of the quadratic equation

$$r^2-Kr+N=0,$$

and $C_1$, $C_2$, $C_3$, $C_4$ are constants, which are to be determined from the initial values of $V_{1n}$ and $V_{2n}$.

The values of the constants $r_1$, $r_2$, $C_1$, $C_2$, $C_3$, $C_4$, L, M, R, S, obtained by actual solution, are given in Table I.

TABLE I.

$$L= \quad S=(a_{25,25}\,a_{25,24}-a_{25,13}\,a_{25,16})\,\Delta,$$

$$-M=-R=(a_{25,25}\,a_{25,16}-a_{25,13}\,a_{25,24})/\Delta,$$

$$\Delta\equiv(a_{25,25}\,a_{25,25}-a_{25,13}\,a_{25,13}).$$

$$C_1= \quad C_3=(V_{11}+V_{21})/2r_1,$$

$$C_2=-C_4=(V_{11}-V_{21})/2r_2.$$

$$r_1=(a_{25,24}+a_{25,16})/(a_{25,25}+a_{25,13}),$$

$$r_2=(a_{25,24}-a_{25,16})/(a_{25,25}-a_{25,13}).$$

$$r_2-L=M=R=r_2-S,$$

$$r_1-L=-M=-R=r_1-S,$$

$$r_2-r_1=2M=2R.$$

* G. Boole, 'A Treatise on the Calculus of Finite Differences,' Chapter 11.

The notation employed in this table requires a word of explanation. It will be noted that expressions of the form $a_{pq,rs}$ occur. These are defined by $a_{pq,rs}=a_{pr}+a_{ps}+a_{qr}+a_{qs}$.

The physical significance of these expressions, as Maxwell * has pointed out, is that they are the electrical coefficients of pairs of bodies: for example, $a_{25,25}$ represents the coefficient of capacity of the bodies 2 and 5, in this case *connected together* ; $a_{25,13}$ the coefficient of induction of the pair 25 and the pair 13 ; etc.

Selecting $r_1$ for further analysis, it can be proved that

$$-1<r_1<+1.$$

In order to show that $r_1$ is less than $+1$, let us write it in the form

$$r_1=\cfrac{1+\cfrac{a_{25,6}+a_{25,4}}{a_{25,1}+a_{25,2}}}{1+\cfrac{a_{25,3}+a_{25,5}}{a_{25,1}+a_{25,2}}}.$$

Now we know from the properties of the coefficients that

$$\left.\begin{aligned} &a_{25,6}+a_{25,4}<0,\\ &a_{25,1}+a_{25,2}>0,\\ &a_{25,3}+a_{25,5}>0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence $r_1$ must be less than unity.

The first of the inequalities (7) follows directly from the fact that every coefficient of induction is negative. The second of the inequalities can be proved as follows :—

$$a_{25,1}+a_{25,2}=a_{21}+a_{51}+a_{22}+a_{52}\ ;$$

but, since the machine is symmetrical, $a_{51}=a_{23}$, so that our expression becomes

$$a_{25,1}+a_{25,2}=a_{22}+a_{21}+a_{23}+a_{25}\ ;$$

and this is positive, since

$$a_{22}>|a_{21}+a_{23}+a_{25}|.$$

The third inequality can be established in a similar manner, for $(a_{25,3}+a_{25,5})$ can be written in the form

$$\begin{aligned} a_{25,3}+a_{25,5}&=a_{23}+a_{53}+a_{25}+a_{55}\\ &=a_{55}+a_{15}+a_{25}+a_{35}\ ; \end{aligned}$$

* J. C. Maxwell, 'A Treatise on Electricity and Magnetism,' i. p. 110, 3rd ed. (1892).

and this is positive, since

$$a_{55} > | a_{15} + a_{25} + a_{35} | .$$

In order to show that $r_1$ is greater than $-1$, we proceed as follows :—If $r_1$ is less than or equal to $-1$, we have

$$\frac{a_{25,16} + a_{25,24}}{a_{25,13} + a_{25,25}} \leqq -1 ;$$

that is, $$a_{25,16} + a_{25,24} + a_{25,13} + a_{25,25} \leqq 0.$$

If this inequality is to be satisfied, we must have

$$A + B + C \leqq 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $$A = a_{22} + a_{21} + a_{23} + a_{24} + a_{25} + a_{26},$$

$$B = a_{22} + a_{21} + a_{23} + a_{25},$$

$$C = a_{55} + a_{15} + a_{25} + a_{35} + a_{45} + a_{65} ;$$

but all the quantities A, B, C are positive and greater than zero, so that the inequality (8) cannot be true. Hence $r_1$ cannot be less than or equal to $-1$.

A similar analysis shows that $r_2$ lies in the range

$$-1 < r_2 < +\infty .$$

Summarizing the results obtained so far, we have

$$\left.\begin{aligned} V_{1n} &= (V_{11} + V_{21}) r_1^n / 2r_1 + (V_{11} - V_{21}) r_2^n / 2r_2, \\ V_{2n} &= (V_{11} + V_{21}) r_1^n / 2r_1 - (V_{11} - V_{21}) r_2^n / 2r_2, \\ n &= 1, 2, 3, \ldots . \\ -1 < r_1 < +1, &\qquad -1 < r_2 < +\infty . \end{aligned}\right\} . \quad (10)$$

From equations (10) a number of important conclusions can be drawn :—

First, it is seen that the terms in $r_1^n$ represent *transient terms*, which vanish for large values of $n$ and are absent altogether, provided the potentials of the inductors are equal and opposite to begin with.

Secondly, if $r_2 > 1$, and this is the case in practice, the potentials of the inductors eventually become equal and opposite, and approach infinity as $n$ approaches infinity.

Thirdly, for the case $r_2 > 1$, $r_1 < 0$ we have an *oscillatory*

transient term, since the potential of the inductor then satisfies the equation

$$V_{1n} = C_1(-\rho_1)^n + C_2 r_2^n, \qquad \rho_1 \equiv |\, r \,|,$$

$(-\rho_1)^n$ being alternately positive and negative.

Finally, it is evident that $r_2$ is the ratio by which the potential of the inductors increases per quarter revolution.

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XXVIII. *On the Incident and Emergent Velocities of Photo-Electrons emitted from Thin Platinum Films.* *By* Chien Cha, *Ph.D.* *

*Introduction.*

It is well known that Röntgen rays and $\gamma$ rays, on passing through thin films, produce different ionization effects on the two sides. A similar asymmetry has been observed in the numbers and velocities of photo-electrons emitted from the two sides of a thin film. The present work deals with the velocities of these electrons only. The side of a thin film upon which light falls will be referred to as the incident side, and the other side will be referred to as the emergent side.

J. Robinson † investigated the velocities of photo-electrons emitted by a thin platinum film on the incident and on the emergent side. He found that for very thin films (less than $10^{-7}$ cm.) the maximum velocity of emission due to emergent light was greater than that due to the incident; for thicker films the reverse was found to be true. In his first paper he gave 1·24 and in his second paper 1·12 as the maximum value for the ratio of emergent to the incident velocities. The same question was taken up by O. Stuhlmann, Jr. ‡ at a later date. He confirmed Robinson's results except in so far as the ratios given by Stuhlmann were a little larger.

Neither the electromagnetic theory of light nor the quantum theory indicates any asymmetry of a measurable order in the photo-electric effect. Richardson § and Swann ‖ independently investigated the matter from the quantum

* Communicated by Prof. W. F. G. Swann. Presented at the Christmas meeting of the American Physical Society, December 1923.

† Phil. Mag. xxiii. p. 542 (1912); Pt. II., Phil. Mag. xxv. p. 115 (1913).

‡ Phys. Rev. iv. p. 195 (1914).

§ Phil. Mag. xxv. p. 144 (1913).

‖ Phil. Mag. xxv. p. 534 (1913).

theory in slightly different forms, and Swann also investigated it from the classical electromagnetic theory. In all cases, the order of magnitude of the asymmetry predicted is far below that of the asymmetry actually observed.

In the works of Robinson and Stuhlmann cited above, the films used were produced by means of cathode sputtering. Also, the ultra-violet light was furnished by a quartz mercury lamp and was unresolved. The results thus obtained could not be used to decide whether or not the effect under investigation has any fundamental significance, because it is quite conceivable that the conditions under which the experiments were carried out might have impressed upon the results this observed peculiarity. It would then seem to be in the interest of more precise information to have the phenomenon re-investigated with unresolved light as well as monochromatic light, with sputtered films as well as evaporated films.

### *Experimental Arrangement and Method.*

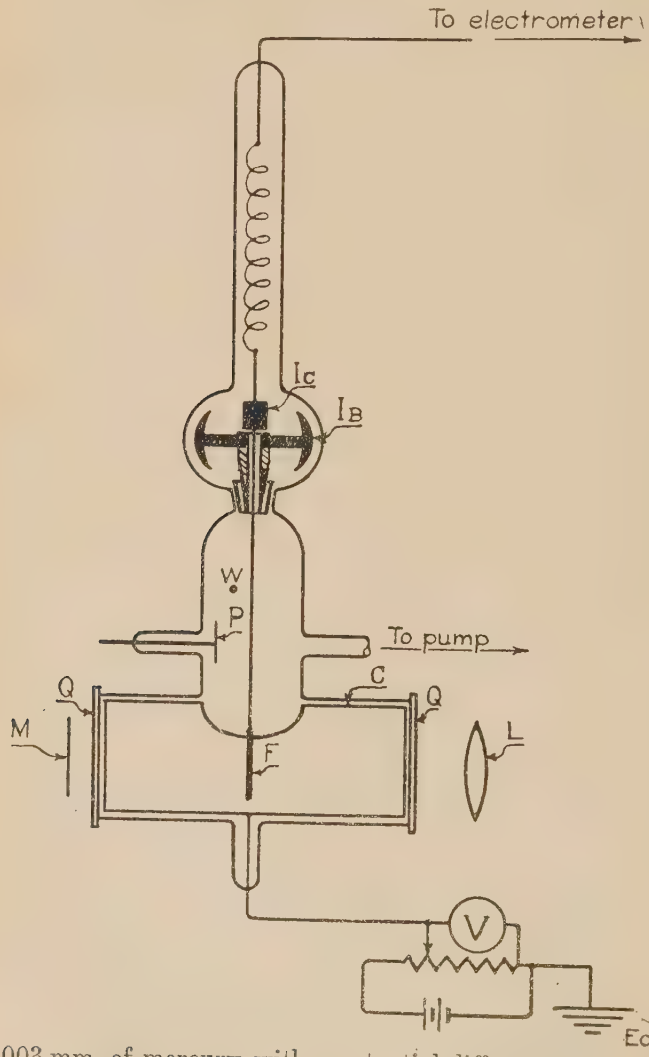
Figure 1 gives a diagrammatic representation of the apparatus used in the present experiment. It was made of glass with two quartz windows Q for the admission of ultra-violet light. F is a quartz plate upon which platinum films were deposited. P is a platinum plate for the purpose of sputtering, and W a platinum wire for the purpose of depositing films by evaporation. $I_C$ is an iron cylinder. $I_B$ are two iron bars with hemispherical attachments at the ends. The iron cylinder and bars were operated by electromagnets not shown in the figure for the purpose of lifting and turning the plate F respectively. C is an aluminium cylinder, the potential of which could be altered by means of a potentiometer system. L is a condensing quartz lens. M is a mirror silvered on the surface.

The process of obtaining an evaporated film was as follows :—The wire W was heated to a white heat by passing a current through it. Then the quartz plate F was lifted into a position directly opposite the plate P by means of an electromagnet operating on the iron cylinder I. The electromagnet, being made to move up and down along the upper portion of the tube by the rotation of an eccentric cam, in turn caused the plate to move up and down in front of the hot wire while evaporation was going on. This vertical motion of the plate extending over a distance of about 7 cm. was necessary in order to make sure of the uniformity of the resulting film.

To obtain a sputtered film, the plate P was connected to the negative pole of a high tension battery, the cylinder C

was connected to the positive pole and the quartz plate F was lifted into a position directly opposite the plate P. All sputtered films used were obtained at a pressure of

Fig. 1.




·003 mm. of mercury, with a potential difference of 900 volts and a current of 20 milliamperes.

A quartz mercury lamp was used as the source of light. When unresolved light was used, a narrow beam of it was

allowed to fall on the quartz lens L which focussed it through the film on the mirror M. The latter in turn reflected the light back on to the film. With this arrangement, the images on the film due to the primary and the reflected beam were of the same size. Monochromatic light was obtained by means of a Hilger monochromatic illuminator. It was focussed as in the case of unresolved light.

Measurements of the maximum potential to which the film was raised as a consequence of the emission of photo-electrons were made by applying a negative potential to the aluminium cylinder C such that only potential less than 1/100 of a volt was left to be indicated by the deflexion of the electrometer. The reading of a voltmeter included in the potentiometer circuit, added to the deflexion of the electrometer in terms of volts, gave the true potential of the film. In each reading taken, light was first shut off from the mirror M until the electrometer had attained its full deflexion and then was allowed to fall on the mirror. Any consequent change in the deflexion was a measure of the difference of the incident and emergent velocities of electrons emitted from the film. The film was then turned through an angle of 180° by means of an electromagnet operating on the iron bars I so that the quartz side was facing the lens L, and a similar reading was taken.

A liquid air vacuum was maintained in the apparatus throughout the course of the experiment except when sputtered films were being made. Evaporated films were made and kept in liquid air vacuum.

### *Results with Evaporated Films.*

It is customary to express the velocity of an electron in terms of the potential that has to be applied in order to stop its motion. The velocity $v$ and the potential V are connected by the relation :

$$\tfrac{1}{2}mv^2 = e\mathrm{V},$$

where $m$ is the mass and $e$ the charge of the electron.

Table I.* gives the average readings obtained with monochromatic light, for different wave-lengths. All the data in this table refer to cases where the primary beams of light fell upon the film side of the plate. They clearly show that the asymmetry of velocities, which was looked for in the experiment, is absent if the film is obtained by evaporation.

* The apparent absolute agreement of the readings for mirror on and mirror off simply indicates that the mirror caused an alteration of the electrometer deflexion corresponding to less than 0·001 volt in each case.

Table I.

(Monochromatic Light: Evaporated Films.)

| Time of Deposit (in min.). | Potentials measuring Maximum Emission Velocities (in volts). | | | | | |
|---|---|---|---|---|---|---|
| | $\lambda=2537$Å. | | $\lambda=2150$Å. | | $\lambda=1849$Å. | |
| | Mirror off. | Mirror on. | Mirror off. | Mirror on. | Mirror off. | Mirror on. |
| 3 ..... | 0·510 | 0·510 | 1·423 | 1·423 | 2·281 | 2·281 |
| 4·5 ..... | 0·508 | 0·508 | 1·420 | 1·420 | 2·282 | 2·282 |
| 6 ..... | 0·511 | 0·511 | 1·419 | 1·419 | 2·280 | 2·280 |

Furthermore, these readings show that this absence of asymmetry is independent of the thickness of film and the wave-length of light over the range covered by the experiment.

When this result was first obtained, it was feared that the amount of light reflected back on to the film by the mirror M, which was supposed to be responsible for the emergent photo-electric effect, might not have been sufficient to give rise to any measurable photo-electric effect. In other words, it was feared that the intensity of light might have been very greatly reduced by the reflexion, and absorption in the film. Therefore a set of readings of the photo-electric current was taken after every set of readings of the photo-electric potentials. The current readings corresponding to the potential readings of Table I. are given in Table II. In the latter table, the current reading obtained when the mirror M was shut off is taken to be unity for each particular thickness of film and also for each particular wave-length of light. The readings clearly show that the amount of light reflected back on to the film in each case is amply sufficient for the production of a photo-electric effect. Consequently, if the emergent velocity were greater than the incident velocity, this effect should have manifested itself in the measurements of Table I.

Table II.

(Monochromatic Light: Evaporated Films.)

| Time of Deposit (in min.). | Relative Magnitude of the Photo-Electric Current. | | | | | |
|---|---|---|---|---|---|---|
| | $\lambda=2537$Å. | | $\lambda=2150$Å. | | $\lambda=1849$Å. | |
| | Mirror off. | Mirror on. | Mirror off. | Mirror on. | Mirror off. | Mirror on. |
| 3 ..... | 1 | 1·65 | 1 | 1·64 | 1 | 1·62 |
| 4·5 ..... | 1 | 1·52 | 1 | 1·49 | 1 | 1·50 |
| 6 ..... | 1 | 1·36 | 1 | 1·35 | 1 | 1·36 |

There is, however, left another possibility. We remarked in the earlier part of this paper that Robinson found, for films

thicker than $10^{-7}$ cm., the ratio of emergent to incident velocities to be less than unity. Owing to the particular manner in which the readings recorded in Table I. were obtained, any emergent velocity, if smaller than the incident velocity, would not have a chance to influence the electric potential measured. Although the thickness of the films used was decidedly less than $10^{-7}$ cm. and so could not give the ratio of emergent to incident velocity less than unity according to Robinson's results, nevertheless it was thought desirable to repeat the experiment in such a way as to remove definitely even this improbable possibility. The film was consequently turned through an angle of 180° so that the quartz side was facing the condensing lens, and a set of readings similar to those of Table I. was taken. These readings are given in Table III., and they are identical with those in Table I. within the limit of observational error.

TABLE III.

(Monochromatic Light: Evaporated Films.)

| Time of Deposit (in min.). | Potentials measuring Maximum Emission Velocities (in volts). | | | | | |
|---|---|---|---|---|---|---|
| | $\lambda=2537$ Å. | | $\lambda=2150$ Å. | | $\lambda=1849$ Å. | |
| | Mirror off. | Mirror on. | Mirror off. | Mirror on. | Mirror off. | Mirror on. |
| 3 ..... | 0·509 | 0·509 | 1·421 | 1·421 | 2·279 | 2·279 |
| 4·5 ..... | 0·513 | 0·513 | 1·418 | 1·418 | 2·282 | 2·282 |
| 6 ..... | 0·512 | 0·512 | 1·422 | 1·422 | 2·284 | 2·284 |

In Table IV. are recorded the measurements made with evaporated films when unresolved light was used. Since the

TABLE IV.

(Unresolved Light: Evaporated Films.)

| Time of Deposit (in min.). | Potentials measuring Maximum Emission Velocities (in volts). | |
|---|---|---|
| | Mirror off. | Mirror on. |
| 3 ..... | (2·281) | (2·281) |
| 4·5 ..... | (2·282) | (2·282) |
| 6 ..... | 2·280 | 2·280 |
| 7·5 ..... | 2·278 | 2·278 |

line of shortest wave-length present in the light used was probably 1849 Å, these readings really give the potentials due to that line. The readings enclosed in parentheses were in fact taken with the line 1849 A°. The excellent agreement which they show with the readings actually obtained with unresolved light testifies to the soundness of our statement above. Just as the use of the line 1849 Å did not give rise

to any lack of symmetry in the velocities of the photo-electrons, so the use of unresolved light did not disclose the effect which was looked for in the experiment.

*Calculation of Planck's Constant "h."*

Although the purpose of the present investigation was not the determination of Planck's constant, it was thought to be interesting to find out whether or not the correct value of "$h$" could be calculated from the results obtained. This procedure would serve as a good test of the reliability of the experiment.

The photo-electric effect was among the first effects to which the quantum theory was applied. The following relation connecting the maximum emission energy of a photo-electron and the frequency of the light causing its emission was proposed by Einstein in 1905 and is known as Einstein's equation :

$$Ve = h\nu - w.$$

where $e$ is the electronic charge, $\nu$ the frequency of light, $h$ Planck's constant, V the difference of potential necessary to stop the electron, and $w$ the energy lost in getting out of the substance. Stated in words, this means that the energy of a photo-electron ejected from a substance by light of frequency $\nu$ is equal to the energy associated with a quantum of light of that frequency, namely, $h\nu$, less the loss of energy in getting out of the substance. There are several important implications of the equation, and two of them can be stated as follows:

(1) The energy of a photo-electron is a linear function of the frequency of light.

(2) The slope of the line connecting the energy and the frequency is the well-known quantum constant "$h$."

For a time it was thought by some investigators that the photo-electrons from thin films had necessarily high emission velocities. Stuhlmann and Compton * found that there was no departure from Einstein's law, provided adequate care were taken to prevent the formation of charged layers, apparently due to the presence of soft wax or grease inside the apparatus where the film was sputtered.

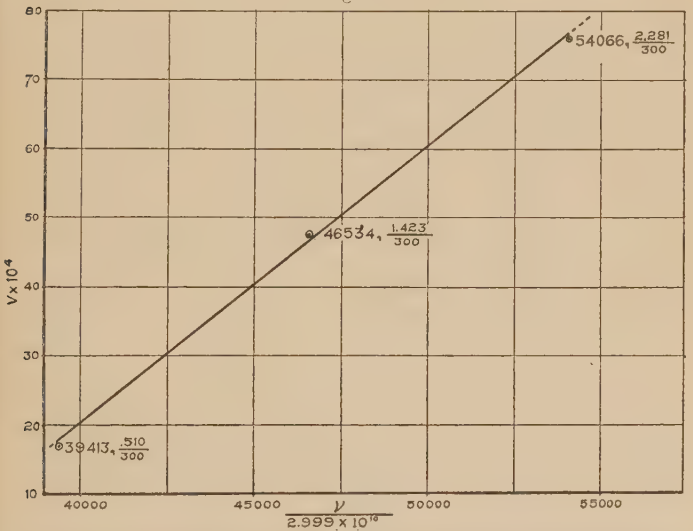
At the time when Robinson and Stuhlmann published their result that the maximum emission energy of the photo-electrons from a thin film was greater when they left the emergent side, the investigations of the emission energies of photo-electrons from ordinary surfaces all led to values of "$h$" about 10 per cent. or 20 per cent. too low. It was therefore suggested that the correct value for "$h$" might be

* Phys. Rev. ii. p. 199 (1913) ; ii. p. 2327 (1913).

obtained if the velocities were measured for the emergent side. In view of the later experiment by Millikan * on the incident velocities of the photo-electrons from which "$h$" was accurately obtained, the above explanation can no longer be maintained.

The curve in fig. 2 was plotted with the readings recorded in Table I. for the film whose time of deposit is 3 minutes. Since the readings for the other films are practically the same as those used, the curves due to them are not included here. It is clear that, over the range covered by the present experiment, the maximum emission energy of a photo-electron emitted from a thin evaporated film is a linear function of the frequency of light.

Fig. 2.




The curve in fig. 2 was plotted with the maximum potential in e.s.u. against the frequency of light. If Einstein's equation holds in this case, the slope of the line should be equal to $h/e$, where $h$ denotes Planck's constant and $e$ denotes the electronic charge. The value of this slope can of course be measured directly from the graph. But it was thought desirable to obtain it in a little different way. A straight line was drawn through every possible pair of points and the slope of the line was calculated. The mean of these slopes was then taken to be the slope of the line representing all the points. Its value is $\frac{403}{2{\cdot}999} \times 10^{-19}$. Taking

* Phys. Rev. vii. p. 355 (1916).

Millikan's value of $e$, which is $4{\cdot}774 \times 10^{-10}$ e.s.u., we obtain, for the value of Planck's constant,

$$h = \frac{403}{2{\cdot}999} \times 10^{-19} \times 4{\cdot}774 \times 10^{-10} \text{ erg. sec.}$$
$$= 6{\cdot}415 \times 10^{-27} \text{ erg. sec.}$$

Comparing this with the value of "$h$" given by Millikan, namely, $6{\cdot}547 \times 10^{-27}$, the value calculated from the present experiment is about 2 per cent. too low. In view of the fact that the experimental error is probably of this order of magnitude, it is believed that no better agreement can be expected.

*Results with Sputtered Films.*

In Table V. are recorded the potentials corresponding to the maximum emission velocities of photo-electrons from sputtered films when monochromatic light was used. Here appears distinctly the asymmetry of emission velocities which was absent in the case of evaporated films. Two other points are also noticeable. First, over the range of the present experiment, the ratio E/I decreases with the increase of the thickness of the film. The form of the curve representing

Table V.—(Monochromatic Light: Sputtered Films.)

| | | | | |
|---|---|---|---|---|
| Time of Deposit (in sec.) | | 30 | 60 | 90 |
| Potentials Measuring Maximum Emission Velocities (in volts). 2537 Å. | Mirror off (I) | 0·524 | 0·553 | 0·585 |
| | Mirror on (E) | 0·630 | 0·634 | 0·640 |
| | E/I | *1·20* | *1·15* | *1·09* |
| 2150 Å. | Mirror off (I) | 1·412 | 1·455 | 1·524 |
| | Mirror on (E) | 1·669 | 1·654 | 1·630 |
| | E/I | *1·18* | *1·14* | *1·07* |
| 1849 Å. | Mirror off (I) | 1·915 | 1·984 | 2·077 |
| | Mirror on (E) | 2·278 | 2·265 | 2·271 |
| | E/I | *1·19* | *1·14* | *1·09* |

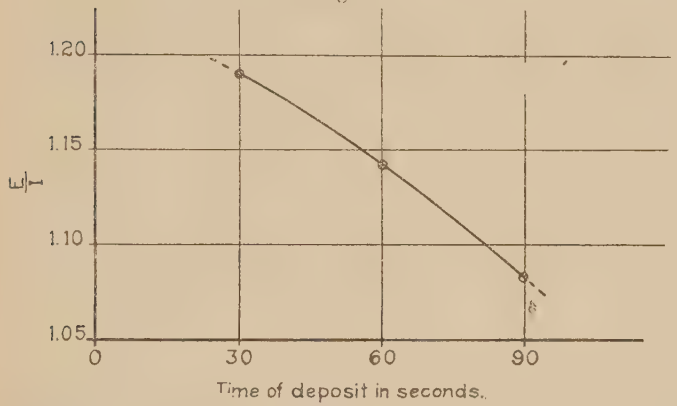
this variation is given in fig. 3. Secondly, over the range of wave-lengths used in the experiment, the ratio E/I stays practically constant.

Table VI. gives the potentials corresponding to the maximum emission velocities of photo-electrons obtained with unresolved light. As in the case of evaporated films, the readings due to the line 1849 Å are here tabulated with the readings due to unresolved light, it being assumed that the line 1849 Å was probably the line of shortest wave-length present in the unresolved light used, and consequently the line responsible for the resulting potential of the film. Here, again is observed a distinct asymmetry of velocities. Also,

TABLE VI.—(Unresolved Light : Sputtered Films.)

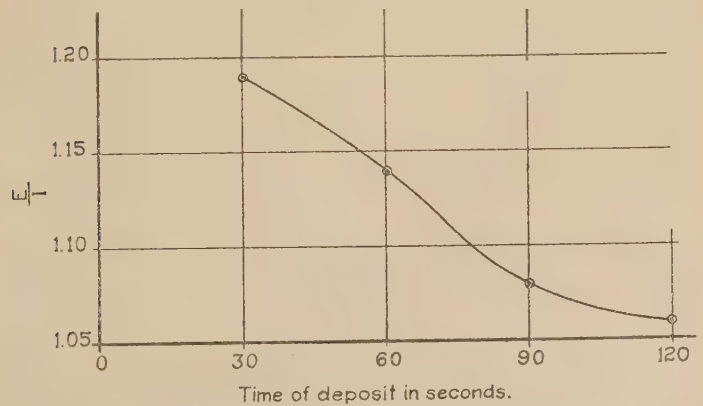
| Time of Deposit (in sec.). | Potentials measuring Maximum Emission Velocities (in volts). | | |
|---|---|---|---|
| | Mirror off (I). | Mirror on (E). | E/I. |
| 30 ..... | (1·915) | (2·278) | (*1·19*) |
| 60 ..... | (1·984) | (2·265) | (*1·14*) |
| 90 ..... | 2·105 | 2·273 | *1·08* |
| 120 ..... | 2·142 | 2·271 | *1·06* |

Fig. 3.




the ratio E/I decreases with the thickness of the film. The form of this variation is shown in fig. 4.

Fig. 4.




It may be mentioned in this connexion that the form of the curves in figures 3 and 4 representing the variation of the ratio of the emergent to incident velocity with the thickness of film is in accord with the results given by Robinson and Stuhlmann.

*Discussion.*

All the data presented in this paper point unmistakably to

the conclusion that the observed asymmetry of velocities of photo-electrons is due to some fundamental difference or differences existing between a sputtered film and an evaporated film. A more thorough study of the properties of these two types of films than has so far been made, will undoubtedly be of great value to the formulation of an adequate explanation. One thing, however, is clear, that is, the fact that while evaporation is done in the best vacuum possible, sputtering requires a fairly high pressure. This immediately leads to the possibility that the residual gases which are occluded or absorbed by the particles of matter during the process of sputtering may be chiefly responsible for the subsequent lack of symmetry which is observed on the maximum velocities of the photo-electrons emitted by the film. If in addition we make the assumption that photo-electrons are shot out from their systems in the direction of the light, it seems obvious that electrons due to the incident light, in order to come out on the film side, will have to describe orbits within the film, thus travelling a greater distance through the film and consequently through the occluded gases than electrons due to the emergent light. Just what the electrons do when they travel through the occluded gas is not known, but it is reasonable to suppose that they suffer a loss of velocity. This would account for the excess of the emergent over the incident velocity.

It is of course not practicable to base any conclusions on a comparison of the absolute velocities of the electrons emitted from evaporated and sputtered films, as the work function may be quite different for the two cases.

A rather remarkable result is found if we plot the maximum velocity against the frequency for sputtered films. Instead of obtaining a linear relation as in the case of evaporated films, we obtain curves departing far from linearity, so that it would be impossible to attach any significance to the calculation of "$h$" from the results for sputtered films. The non-linearity of the curves suggests that both in the case of the electrons emitted by incident light and also in that of the electrons emitted by emergent light, the loss of energy in passing through the occluded gases in the process of escape from the surface is a function of the velocity of the electron.

The investigation described in this paper was carried out under the direction of Professor W. F. G. Swann, at the University of Minnesota, and the writer desires to express his deep appreciation to Professor Swann for the very kind advice and encouragement which he extended throughout the course of the work.

Department of Physics,
University of Minnesota.

## XXIX. *A Gyro Compass incorporating two Gyroscopes.* *By* Professor Sir James B. Henderson, *D.Sc.**

THE gyroscopic compass is a scientific instrument of the most sensitive kind and is an excellent example of the application of theory in practice. It can be so designed that the action of the compass under all conditions can be calculated to a high degree of accuracy so that before the instrument is constructed its behaviour under any circumstances can be foretold, and when the design has materialised the problem then becomes one of locating and eliminating the causes of any differences between the actual and the theoretical behaviour.

During the last ten years the most important problem in gyro compasses has been the elimination of the deviation of the compass introduced by the oscillations of the ship due to wave motion; and when the history of this period comes to be written, an important phase of development will be that in which the compass comprised two gyroscopes. One such compass made by the Sperry Gyroscope Company had one of the gyroscopes meridian-seeking and the other the reverse, but the latter was compelled by discontinuous torques to follow the former. In another two-gyro compass made by the author both gyroscopes were meridian-seeking, all torques being continuous functions and the interaction of the two gyroscopes providing mutual damping. The mathematical analysis of the motion of this latter compass, the conditions for its stability, and the investigation of the damping coefficients of both periods form the subject of this paper.

### *Description of the Compass.*

Fig. 1 represents diagrammatically a front elevation of the compass without its supports or following motor, and fig. 2 represents a side elevation of the sensitive element alone.

The compass contains two gyroscopes A and A′, supported respectively upon horizontal trunnion axes B and B′ in vertical gimbal rings C and C′ pivoted on the vertical trunnions V and V′ in the frame D. This frame is pivoted on the vertical trunnion U in the follower Z. The follower is supported in a binnacle on a large ball race by

* Communicated by the Author.

the disk E which is toothed on its periphery and is driven in azimuth by a small motor controlled by a roller contact F carried by the frame D of the sensitive element and engaging with a two-part commutator G fixed to the follower Z. The following motor therefore keeps the follower Z in phase in azimuth with the roller contact F. The frame D is suspended from the bridge R carried by the plate E by the filar suspension S ; the ring C is suspended from the

Fig. 1.

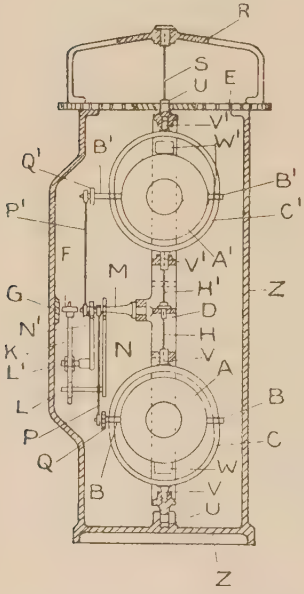
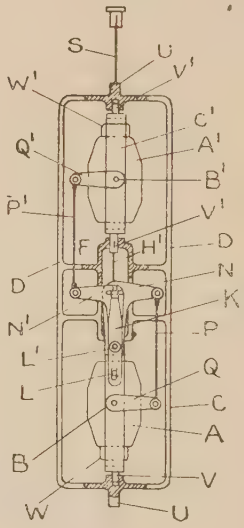



Fig. 2.




frame D by the torsion element H, and the ring C′ is supported from the frame D by the torsion element H′ which is in thrust.

The gyroscope A is put in gravitational stability by the weight W, and the gyroscope A′ is put into gravitational instability by the equal weight W′. The main idea in the design is to get an arrangement in which the gravitational control is practically neutral for periodic accelerations of short period due to waves, which is the case if the gyroscope cases A and A′ carrying the weights W and W′ are

connected together by a viscous element. The details of this arrangement are omitted in the diagram in order to prevent complication. The gyroscope A′, if acting alone, oscillates about the meridian with its axis of angular momentum pointing South, and in order to damp these oscillations energy has to be supplied to the gyroscope by couples about the vertical axis *assisting* the precession. The gyroscope A, if acting alone, oscillates about the meridian with its axis of angular momentum pointing North, and to damp these oscillations energy has to be taken from the gyroscope by couples acting about the vertical axis *opposing* the precession. Hence a mutual torque between the two gyroscopes due to the torsion elements H and H′ will assist in damping the oscillations of both gyroscopes if the stable gyroscope A precesses faster than the unstable one A′, *i. e.* if A has smaller angular momentum than A′.

With such a combination there would evidently be a stable state with A and A′ in the same azimuth and therefore producing no twist in the filar H–H′ and with A′ tilted more than A so that both precessed in azimuth at the same rate, in which condition they might precess together continuously round and round in azimuth. In order to prevent such a stable state from arising, an external torque with a vertical axis is applied to both gyroscopes proportional to the difference between the tilts of the two rotor axes and in such a direction as to tend to keep the axes of A and A′ in phase as regards tilt. This torque is produced by displacing the roller contact F in azimuth by an amount proportional to the difference in tilt between the two gyros. Horizontal cranks Q and Q′ are fixed to the trunnions B and B′ of the gyros A and A′ (fig. 2), and these are connected by connecting rods P and P′ with two bell-cranks N and N′ pivoted on a spindle M carried by the frame D, so that the bell-cranks N and N′ tilt in phase with the two gyros. The roller contact F is mounted on a lever K which is pivoted on a crank-pin L′ fixed to the bell-crank N′, and a crank-pin L on the bell-crank N engages with a slot in the end of the lever K. It will be evident that this mechanism displaces the roller F in azimuth by an amount proportional to the difference of tilt of the two gyros, and by actuating the following motor twists the suspension S by an amount proportional to the difference in tilt of the two gyroscopes in the direction required to reduce that difference.

*Equations of Motion.*

Let $\theta_1$ be the azimuthal deviation of gyro A } reckoned +ive from N. to W.
$\theta_2$ „ „ „ gyro A′ }
$\phi_1$ the tilt of gyro A reckoned +ive with N. end high.
$\phi_2$ „ gyro A′ „ „ „
$G\phi_1$ the gravitation torque on gyro A due to tilt $\phi_1$.
$G\phi_2$ „ „ „ gyro A′ „ $\phi_2$.
$h(\theta_1-\theta_2)$ mutual torque due to torsion element H–H′.
$n(\phi_1-\phi_2)$ torque on both gyros applied externally through suspension.

$I_1\Omega_1$ angular momentum of gyro A.
$I_2\Omega_2$ „ „ „ gyro A′.
$\omega_1$ angular velocity of precession of gyro A } due to unit couple.
$\omega_2$ „ „ „ „ gyro A′ }
$\omega$ angular velocity of the earth.
$\lambda$ latitude.

$\omega_1$ and $\omega_2$ are introduced to avoid reciprocals in the equations.

$$\omega_1=\frac{\text{unit couple}}{I_1\Omega_1} \quad \text{and} \quad \omega_2=\frac{\text{unit couple}}{I_2\Omega_2}.$$

It is most convenient to refer the motion of the gyroscopes to axes which move with the rings C and C′. Thus in figs. 3 and 4, which refer respectively to the gyroscopes A and A′, OB represents the horizontal trunnion axis B, OV represents the vertical trunnion axis V, and OX is perpendicular to both and represents the azimuth of the rotor axis OA. OM represents the meridian, and OP is a vector parallel to the axis of the earth and represents the angular velocity of the earth. The angle MOX is $\theta_1$, AOX is $\phi_1$, and POM is $\lambda$ the latitude.

Writing down on each axis the component angular momentum, the component couple, and the component precessional velocity, and equating the couples to the rate of increase of the component angular momentum, assuming $\theta$ and $\phi$ to be small angles, we get from fig. 3 :

$$G\phi_1 = I_1\Omega_1(\omega\sin\lambda+\dot{\theta}_1)-I_1\Omega_1\phi_1\omega\cos\lambda,$$

$$h(\theta_2-\theta_1)+n(\phi_2-\phi_1) = I_1\Omega_1\dot{\phi}_1+I_1\Omega_1\omega\cos\lambda\,.\,\theta_1;$$

Fig. 3.

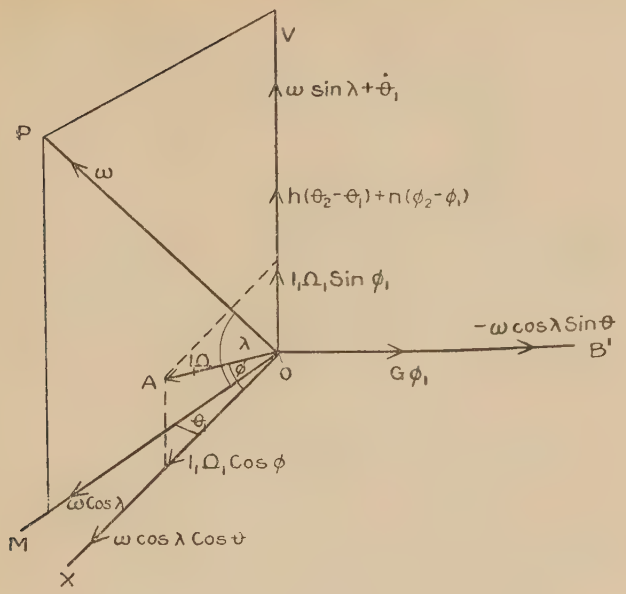
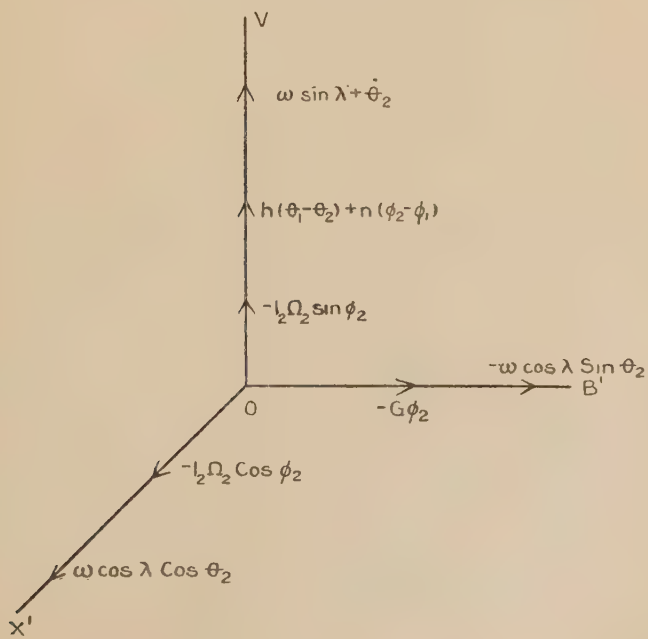

V
ω sin λ + θ̇1
P
ω
h(θ2 − θ1) + n(φ2 − φ1)
I1Ω1 Sin φ1
−ω cos λ Sin θ
B'
λ
Ω
φ
A
O
G φ1
θ1
I1Ω1 Cos φ
ω Cos λ
M
ω cos λ Cos θ
X

Fig. 4.


V
ω sin λ + θ̇2
h(θ1 − θ2) + n(φ2 − φ1)
−I2Ω2 sin φ2
−ω cos λ Sin θ2
B'
O
−Gφ2
−I2Ω2 Cos φ2
ω cos λ Cos θ2
X'

and from fig. 4 :

$$-G\phi_2 = -I_2\Omega_2(\omega \sin\lambda + \dot{\theta}_2) + I_2\Omega_2\phi_2\omega\cos\lambda,$$

$$h(\theta_1-\theta_2)+n(\phi_2-\phi_1) = -I_2\Omega_2\dot{\phi}_2 - I_2\Omega_2\,\omega\cos\lambda\,.\,\theta_2.$$

Whence

$$\left.\begin{aligned}
\dot{\theta}_1 &= \phi_1[G\omega_1+\omega\cos\lambda]-\omega\sin\lambda,\\
\dot{\theta}_2 &= \phi_2[G\omega_2+\omega\cos\lambda]-\omega\sin\lambda,\\
\dot{\phi}_1 &= -\omega\cos\lambda\,.\,\theta_1 - h\omega_1(\theta_1-\theta_2)-n\omega_1(\phi_1-\phi_2),\\
\dot{\phi}_2 &= -\omega\cos\lambda\,.\,\theta_2 - h\omega_2(\theta_1-\theta_2)+n\omega_2(\phi_1-\phi_2).
\end{aligned}\right\}. \quad (1)$$

Let

$$\theta_1+\theta_2 = \alpha,\quad \theta_1-\theta_2 = \beta,\quad \phi_1+\phi_2 = \gamma,\quad \text{and}\quad \phi_1-\phi_2 = \delta.$$

Let $\omega_1+\omega_2 = s$ and $\omega_1-\omega_2 = d$,

then by substitution we get

$$\left.\begin{aligned}
\dot{\alpha} &= G\left(\frac{\gamma}{2}s+\frac{\delta}{2}d\right)-2\omega\sin\lambda,\\
\dot{\beta} &= G\left(\frac{\gamma}{2}d+\frac{\delta}{2}s\right),\\
\dot{\gamma} &= -\omega\cos\lambda\,.\,\alpha - hs\beta - nd\delta,\\
\dot{\delta} &= -\omega\cos\lambda\,.\,\beta - hd\beta - ns\delta.
\end{aligned}\right\}. \quad . \quad . \quad . \quad (2)$$

The term $\omega\cos\lambda$ has been omitted in comparison with $G\omega_1$. In the ordinary single-gyro compass $G\omega_1$ is $\frac{2\pi}{3 \text{ minutes}}$ whereas $\omega\cos\lambda$ is $\frac{2\pi}{48 \text{ hours}}$ at $\lambda=60°$, and the latter is negligible in comparison with the former. $\alpha$ is of course double the mean deviation of the two gyros, $\beta$ is their relative motion in azimuth, $\gamma$ is double the mean tilt, and $\delta$ is their relative tilt.

To find the settling position $\dot{\alpha}=\dot{\beta}=\dot{\gamma}=\dot{\delta}=0$ and denote zero value by the suffix $_0$, or more directly in equations (1) let

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\phi}_1 = \dot{\phi}_2 = 0;$$

then

$$ {}_0\phi_2 = \frac{\omega \sin\lambda}{G\omega_2}\ ; \quad {}_0\phi_1 = \frac{\omega \sin\lambda}{G\omega_1}\ ; $$

$$ {}_0\theta_1(h\omega_1 + \omega\cos\lambda) = {}_0\theta_2 \,.\, h\omega_1 - n\omega_1\delta_0 $$

$$ = h\omega_1 \,.\, {}_0\theta_2 + \frac{\omega \sin\lambda \,.\, nd}{G\omega_2}\ : $$

and $$ {}_0\theta_2 = \frac{-2n\tan\lambda}{G}. $$

All these values are due to $\omega \sin\lambda$, and by putting a weight on the North side of the stable gyro and one on the South side of the unstable gyro, all can be reduced to zero as is done in the single-gyro compass.

*Periodic Motion.*

Let $\alpha = \alpha_0 + A\epsilon^{pt}$, $\beta = \beta_0 + B\epsilon^{pt}$, $\gamma = \gamma_0 + C\epsilon^{pt}$, and $\delta = \delta_0 + D\epsilon^{pt}$.

A, B, C, and D are determined by the initial conditions and we have to eliminate these to determine the values of $p$. Substituting in equations (2) and eliminating A, B, C, and D, we get the determinant

$$ \begin{vmatrix} p & 0 & -\frac{Gs}{2} & -\frac{Gd}{2} \\ 0 & p & -\frac{Gd}{2} & -\frac{Gs}{2} \\ \omega\cos\lambda & hs & p & nd \\ 0 & (hd+\omega\cos\lambda) & 0 & (p+ns) \end{vmatrix} = 0. \quad . \quad (3) $$

Expanding this we get

$$ p^4 + p^3 ns + p^2 \,.\, Gs(hd + \omega\cos\lambda) + pnG\frac{s^2-d^2}{2}(hd + \omega\cos\lambda) $$

$$ + \omega\cos\lambda(hd + \omega\cos\lambda) \,.\, G^2\frac{s^2-d^2}{4} = 0\ ; $$

or

$$ p^4 + p^3 ns + p^2 Gs(hd + \omega\cos\lambda) + 2pnG\omega_1\omega_2(hd + \omega\cos\lambda) $$

$$ + \omega\cos\lambda \,.\, hdG^2\omega_1\omega_2 = 0\ ; $$

or if $\omega\cos\lambda$ is small compared with $hd$, we get

$$ p^4 + p^3 ns + p^2 Gshd + 2pnGh\omega_1\omega_2 d $$

$$ + \omega\cos\lambda \,.\, hdG^2\omega_1\omega_2 = 0. \quad . \quad (4) $$

In order that the instrument may be a compass this equation must represent two damped oscillations, one being the oscillation of the two gyros together and the other their relative oscillation. The former is the period of the compass and the latter a much shorter period.

The first condition in order that this equation shall represent two damped oscillations is that all coefficients of $p$ shall be positive *. That is satisfied if $d$ is positive, *i. e.* if $\omega_1 > \omega_2$. That is, the stable gyro has smaller angular momentum (or greater gravity control) than the unstable one. The equation in $p$ has to be resolved into two factors as follows :—

$$(p^2+2A_1p+A_1^2+B_1^2)(p^2+2A_2p+A_2^2+B_2^2) = 0,$$

in which $A_1$ and $A_2$ are the damping coefficients of the two oscillations, and $B_1$ and $B_2$ are the two frequencies or periodicities, *i. e.* $\frac{2\pi}{\text{period}}$.

When the damping is zero let $p_1$ and $p_2$ be the periodicities ; then

$$(p^2+p_1^2)(p^2+p_2^2) = 0 ;$$

or $$p^4+p^2(p_1^2+p_2^2)+p_1^2p_2^2 = 0.$$

If in equation (4) we make $n=0$, the damping disappears and

$$p_1^2+p_2^2 = Gshd, \quad p_1^2p_2^2 = \omega\cos\lambda\,.\,hdG^2\omega_1\omega_2 ;$$

and if $p_1 = 10p_2$ by assumption, then

$$p_1^2 \doteqdot Gshd,$$

and $$p_2^2 \doteqdot \frac{\omega\cos\lambda\,.\,G\omega_1\omega_2}{s}.$$

The second condition that equation (4) shall represent two damped oscillations is

$$p_1^2 > \frac{\text{coeff. of } p}{\text{coeff. of } p^3} > p_2^2,$$

or $$Gshd > 2Ghd\,.\,\frac{\omega_1\omega_2}{s} > \omega\cos\lambda\,.\,G\omega_1\omega_2\frac{1}{s} ;$$

or $$(\omega_1+\omega_2)^2 > \omega_1\omega_2, \quad \text{and} \quad 2h(\omega_1-\omega) > \omega\cos\lambda.$$

* The conditions that an equation $p^4+ap^3+bp^2+cp+d=0$ shall represent two damped oscillations have been stated by Routh as (1) all coefficients must be positive and (2) $abc-c^2-a^2d>0$. This last condition leaves a very large range of choice of the coefficients, which must therefore be decided by other considerations.

The first is necessarily true and the second can be arranged by suitably choosing the torsion element.

The periodicity in the ordinary single-gyro compass is given by

$$p^2 = G\omega_1 . \omega \cos\lambda = \left(\frac{2\pi}{85 \text{ minutes}}\right)^2.$$

Hence we see that if the gravity control of each gyro in the two-gyro compass is to be approximately the same as in the one-gyro compass—which must be the case if the ballistic deflexion is to take the compass on to the virtual meridian,—then the periodicity of the two-gyro compass must be less than that of the single-gyro compass in the ratio of $1 : \sqrt{2}$ ; *i. e.*, the period of the compass is increased in the ratio of $\sqrt{2} : 1$, or instead of being 85 minutes the period will be 120 minutes.

Assume arbitrarily the ratio $p_1 : p_2 = 10$, and also assume a ratio of $\omega_1 : \omega_2$ say 1·1 : that is, make the stable gyro with 0·9 of the angular momentum of the unstable gyro.

$$p_1^2 = 100\, p_2^2,$$

$$G\,shd \doteqdot \frac{100\, G\omega_1\omega_2\omega \cos\lambda}{\omega_1 + \omega_2};$$

or
$$h(\omega_1 - \omega_2) \doteqdot \frac{100\,\omega_1\omega_2}{(\omega_1 + \omega_2)^2}\,\omega \cos\lambda$$

and
$$\omega_1 = 1{\cdot}1\,\omega_2;$$

$$\therefore \quad h\omega_2 \doteqdot 250\,\omega \cos\lambda,$$

or at $\lambda = 51.^\circ$
$$h\omega_2 = \frac{2\pi}{9 \text{ minutes}}.$$

Hence a twist of 1 radian in the torsion element connecting the gyros would make the unstable gyro precess with a speed equal to one revolution in 9 minutes. This determines the stiffness of the torsion element.

It still remains to determine the stiffness of the suspension through which the damping torques are transmitted and to calculate the damping coefficients $A_1$ and $A_2$.

If we write equation (4) as

$$p^4 + ap^3 + bp^2 + cp + d' = 0,$$

the values of $A_1$ and $A_2$, the damping coefficients, are

given by

$$A_1 \doteq \frac{a\left(p_1^2-\frac{c}{a}\right)}{2(p_1^2-p_2^2)} \quad \text{and} \quad A_2 \doteq \frac{a\left(\frac{c}{a}-p_2^2\right)}{2(p_1^2-p_2^2)}.$$

But

$$\frac{\frac{c}{a}}{p_1^2} = 2\,\frac{Ghd\omega_1\omega_2}{Gs^2hd} = \frac{2\omega_1\omega_2}{(\omega_1+\omega_2)^2} = \frac{2\times 1\cdot 1}{(2\cdot 1)^2} = \frac{1}{2}.$$

$$\therefore \qquad A_1 = \frac{a(1-\frac{1}{2})}{2(1-\frac{1}{100})} = \frac{a}{3\cdot 96}$$

$$\text{and} \qquad A_2 = \frac{a(\frac{1}{2}-\frac{1}{100})}{2(1-\frac{1}{100})} = \frac{a}{4\cdot 05}.$$

$A_2$ is the damping coefficient of the period of the compass. Suppose that we damp out the oscillation in one period, say, to 0·2 per cent., *i.e.* $\epsilon^{-2\pi}$; then $A_2 = \frac{2\pi}{120 \text{ minutes}}$ and $a=ns=\frac{2\pi}{30}$; $\therefore\ n\omega_2=\frac{2\pi}{63 \text{ minutes}}$. Hence the twist in the suspension due to one radian difference of tilt will impart a precessional velocity to the unstable gyro equal to one revolution in 30 minutes.

$$A_1 = \frac{2\pi}{30\times 3\cdot 96} = \frac{2\pi}{119 \text{ minutes}}.$$

Hence the short period is practically damped out in the same time as the long period.

It remains only to note that the torque in the suspension *assists* the precession of the unstable gyro when its tilt exceeds that of the stable one, otherwise it resists this precession. That is, if the North end of the unstable gyro is tilted above the North end of the stable one, the twist in the suspension forces the North end of the unstable gyro to the West.

The particular arrangement shown in figs. 1 and 2 has been selected because it makes the mathematical equations symmetrical for the two gyroscopes and introduces the minimum number of terms. The introduction of the viscous connexion between the gyroscopes adds more terms, as also does any other arrangement of the roller contact. This arrangement, however, serves to typify the mathematical methods.

In this connexion I would like to express my indebtedness to my mathematical colleague Professor Wm. Burnside, F.R.S., for valuable assistance in this and many other problems.

## *Summary.*

The following is a summary of the results of the above discussion :—

It is shown how the two periods of oscillation of a structure carrying two gyroscopes such as a gyro-compass may be calculated ; also the conditions for their mutual damping.

The undamped periods of the compass described are 120 and 12 minutes in latitude 51°.

Gravity control is such that $G\omega_1 = \frac{2\pi}{3{\cdot}14 \text{ minutes}}$ ; *i. e.*, if the gyro is tilted 30° from horizontal its axis will describe a cone round the vertical of 60° angle in $3{\cdot}14 \times \sqrt{3}$ minutes $= 5{\cdot}4$ minutes.

The unstable gyro has 10 per cent. greater angular momentum than the stable one.

The torsion element connecting the two gyros about the vertical is such that a twist of 1 radian would give to the unstable gyro a velocity of precession of 1 revolution in 9 minutes.

The suspension and the linkage to the switch producing the torque in the suspension are such that a tilt of one gyro through 1° relatively to the other produces a rate of precession of the unstable gyro of 0·64 degree per minute.

The long-period oscillation is damped to 0·2 per cent. in 120 minutes.

The short-period oscillation is damped to 0·2 per cent. in 120 minutes.

The compass settles on the meridian with both rotor axes horizontal provided that weights are placed on the N. end of the stable and S. end of the unstable gyros sufficient to produce the precession $\omega \sin \lambda$, the velocity of the meridian about the vertical.

## XXX. *Dimensional Analysis.*

*By* Norman Campbell, *Sc.D.*

[A reply to Mr. Buckingham's criticism (July 1924, p. 141) of my paper (March 1924, p. 481).]

I WAS careful to refrain from any direct criticism of Mr. Buckingham's work; but he has misinterpreted my meaning so gravely that I must be more definite.

I did not cast, and would never think of casting, the smallest doubt on the value of Rayleigh's use of the argument from similarity; and I share fully Mr. Buckingham's great admiration of the work of the Aeronautics Staff of the N.P.L., of which I was at one time an insignificant member. But I deny that this highly important work has anything to do with "dimensional analysis," as expounded by Mr. Buckingham and others. The question is not whether the argument is valuable, but what exactly the argument is.

Mr. Buckingham speaks of expounding the argument in text-book fashion. Alas, there are text-books and text-books! If a writer of a text-book on arithmetic devoted all his space to elaborate rules for drawing lines of the right width in the right places among figures entered in perfectly vertical columns, and if he omitted to point out that the operations entered in this beautiful manner must be conducted with an eye to the multiplication table, would he be able to rebut an accusation that the book was worthless by appealing to the great importance of arithmetic in technical physics?

The analogy is exact. The fundamental conception of the argument as used by Rayleigh is similarity. The concept of dimensions may be convenient in applying it; but Rayleigh did not always employ it, and, as I tried to show, it is never necessary. And yet in one of the papers which (I suppose) constitute the text-book * there is not a single mention of the word similarity, or of any synonym, and nothing but the most oblique hints that the systems to be compared must be even geometrically similar. There is no indication of why dimensions are relevant at all. The paper is a mass of elegant irrelevancies from which the one thing needful is carefully excluded.

The insistence on dimensions and the neglect of similarity is the direct cause of the errors to which I drew attention; nothing of physical importance can be derived from the consideration of dimensions alone. In fact, I would go

* E. Buckingham, Phil. Mag. xlii. p. 696 (1921).

further, and maintain that the conception of dimensions is totally useless and, in view of its dangers, should be abandoned. It was introduced originally to describe how the numerical value of a magnitude would change with the units employed in measuring it, on the assumption that the formal constants of the laws defining the magnitudes would always remain unaltered. Experience has shown that the assumption is untrue; physicists often find it convenient to change the formal constants rather than the numerical value. Moreover, units are now so standardized that the results of any change that ever occurs can be and are tabulated in detail; there is no need to consider how they can be deduced. The use of dimensions in checking an equation by examination of the dimensions of its two sides has also proved worthless; for equations which do not satisfy the criterion can be found in the writings of the most eminent physicists. Such discrepancies (usually due to the omission of "$c$") are doubtless due to inadvertence, and are not to be defended. But if the retention of dimensions in all our text-books cannot save us from such errors; if it encourages so greatly the activities of the modern equivalent of circle-squarers and leads to the abuse of one of the most valuable weapons of experimental research; is it worth while to puzzle with it the heads of our students and, incidentally, our own?

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## XXXI. *The Cooling Power of a Stream of very Viscous Liquid.*

*To the Editors of the Philosophical Magazine.*

Gentlemen,—

In a recent paper * I showed the effect of introducing a viscosity term into Boussinesq's treatment of the cooling power of a stream of inviscid fluid. In the analysis the equations of motion etc. were written down and certain substitutions were made for the variables involved. The resulting equations were quite free from terms representing the properties of the fluid etc. except that, in the notation adopted, a $c\nu/k$ group remained. It followed that the heat loss from similar bodies was given by

$$hL/k\theta = F[(v_\infty L/\nu);\ (c\nu/k)]. \quad . \quad . \quad . \quad (17)$$

This equation is true.

In the body of the paper it was indicated that for very viscous liquids the equation may tend to the form

$$hL/k\theta = F(v_\infty L/\nu).$$

* Phil. Mag. xliv. p. 940 (1922).

Unfortunately this is invalid, for it was erroneously supposed that if certain acceleration terms were neglected the $c\nu/k$ group would disappear from the equations. Actually, however, with the substitutions given, it remains as $(c\nu/k)\mathrm{T}'$.

For the amended analysis the correct substitutions can be obtained from those given in equation (5) by writing $k/c$ for $\nu$ and $\mathrm{P}=\rho v_\infty^2(c\nu/k)\Pi$ for $\mathrm{P}=\rho v_\infty^2\Pi$. All properties of the fluid etc. then vanish from the transformed equations (6) and the boundary conditions (8), except that $(k/c\nu)(\mathrm{U}', \mathrm{V}', \mathrm{W}')$ occur in the equations of motion. From the heat flux relation it then follows generally that

$$h\mathrm{L}/k\theta=\mathrm{F}[(v_\infty \mathrm{L}c/k)\ ;\ (c\nu/k)],$$

an equation which is merely (17) in another form.

For very viscous liquids, however, $k/c\nu$ tends to be very small, and if the acceleration terms $(k/c\nu)(\mathrm{U}', \mathrm{V}', \mathrm{W}')$ are neglected we obtain

$$h\mathrm{L}/k\theta=\mathrm{F}(v_\infty \mathrm{L}c/k).$$

It is interesting that this simpler expression is of the same form as that which applies to inviscid fluids, the viscosity $\nu$ being absent. It is in good agreement with the tendency of my experiments on the forced cooling of wires *—for the most viscous liquids used the heat loss term tended to vary as $\mathrm{R}(v_\infty \mathrm{L}c/k)^{0\cdot38}$, where the factor R represents certain residual effects.

I am, Gentlemen,
Yours faithfully,
August 1924. A. H. Davis.

## XXXII. *Notices respecting New Books.*

*The Measurement of Fluid Pressure and Velocity.* By J. R. Pannell. (Arnold & Co. 10*s.* 6*d.*)

THIS volume, which is a notable production of the Aerodynamics Department of the National Physical Laboratory, has a sad history. It is edited by Mr. Frazer, of the Department, and opens with a foreword by Mr. Southwell, the head of the Department. Mr. Pannell was killed in the trial flight of the rigid airship R. 38, after notable work on the speed and general properties of the airship. His notebooks were partially rescued from the wreck, and showed that this work, on his part, had continued almost to the moment of the disaster. He was one of the original members of the Department.

Mr. Frazer has completed a manuscript which he left as the foundation of a treatise on fluid pressures and velocities. To a

* Phil. Mag. xlvii. p. 1057 (1924). The empirical formula given in the summary tends to the form stated, since $m$ and $n$ tend respectively to 0·19 and 0·38.

great extent, we must regard this as a memorial volume. The various subjects discussed comprise pressure-tube instruments, of which a very complete account is contained in the first chapter; various anemometers, to which the second chapter is devoted, and which may in general be described as moving-part anemometers; hot wire anemometers; direction and velocity meters, and manometers. The work is unique in the comprehensive and balanced account which it gives of all these instruments, none of which, in a modern form, finds a place in any text-book. A final chapter examines the flow of fluids in circular pipes, and is rather more theoretical.

We cannot describe the proof-reading as having been entirely satisfactory. For example, it is somewhat of a shock to see Euler referred to on p. 51 as "Eulzer," in relation to a notable constant which he introduced. Such blemishes are too numerous, and perhaps may disappear in a new edition, which, in a country becoming more alert to the practical side of aerodynamics, will no doubt be called for. The fundamental value of the work lies on the side of the practical description of instruments.

*On the direct numerical Calculation of Elliptic Functions and Integrals.* By Prof. L. V. King. (Camb. Univ. Press, 3*s.* 6*d.* net.)

This little work really does fulfil a long-felt want. It is dedicated to the memory of the late Prof. Harkness, of McGill University.

In spite of all the efforts of, for example, Sir George Greenhill, the numerical tabulation of elliptic functions has not advanced much beyond the point where Legendre left it. It is an expensive matter, and funds do not seem available. It is too customary for the mathematician to regard a problem as solved when its answer is expressed in a standard form of elliptic integrals. The physicist, confronted with this, merely gasps, and proceeds to find an alternative solution in his laboratory. Prof. King, who belongs to both categories, refuses to be satisfied with this position, and he explains, in his preface, just how he was led to the present work by the need for accurate values of the coefficients of self- and mutual-induction of coils.

He makes no claim to novelty in formulæ or in methods of computation, but, nevertheless, we find a great deal; and the work is in fact a notable contribution towards the simplification of the necessary calculations. We can only regret that it perforce stops short with the demonstrations of the proper mode of calculating the Tables in a numerical way. Physicists need tables of elliptic functions on a scale commensurate even with tables of logarithms.

*Collected Researches from the National Physical Laboratory.* Vol. xvii. 1922. (H.M. Stationery Office. 17*s.* 6*d.* net.)

This volume follows the usual lines of the series issued by the National Physical Laboratory. All the papers included have, of

course, already been published elsewhere, and the value of the work lies in the fact that it gives an epitome of the published papers of the Staff of the Laboratory ranging over a given time. They are restricted entirely to optical work and its various side-issues, and are taken from the Proceedings of the Physical Society, the Optical Society, and the Philosophical Magazine. The output is very large, and contains notable contributions to the detail of the subject, and more especially to its application. What is usually called "geometrical optics" is in question, and it is to some extent a matter of regret that so little attention is given in the laboratory to the fact that rays of light have wave-properties, so that the more striking refinements of optical methods recently worked out elsewhere depend so much on the phenomena of interference and of diffraction of light. We cannot avoid the feeling that there is some scope in a Department of this type for a broader outlook on optical questions, in so far as future development of accurate method is intended. But we can only congratulate the various authors on the meticulous care which they take to explore every detail which the older methods still lack, in regard to precision of statement and of inference.

*Applied Elasticity.* By J. Prescott. (Longmans, Green & Co. 25*s.* net.)

This is a work which we can recommend, with some confidence, to the engineer or physicist whose mathematical equipment is not too limited. The author, while stating that the work contains little which can not be found in mathematical treatises on the theory of elasticity, expresses his intention of writing a book for the practical engineer. He follows this object by severe limitation of his illustrative examples to those which can lead to the solution of practical problems. In this respect we consider that his choice is, on the whole, excellent. He leaves aside certain matters on which theory and experiment are not quite in line, and confines himself to the matters on which there is no doubt. A practical man who can read the book will certainly trust it, for he will not find anything of importance which seems to contradict his experience.

Approximate methods of solution based on the principle of minimum energy are used to an unusual extent, for a sufficiently valid reason which the author explains, and a reason which, however unsatisfactory to the logical sense, is certainly, in its results, effective in practice. He thus avoids much tedious analysis on important problems, especially those relating to questions of stability, and of normal elastic oscillations. The book contains also a certain amount of original matter due to the author, and not hitherto published.

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[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]